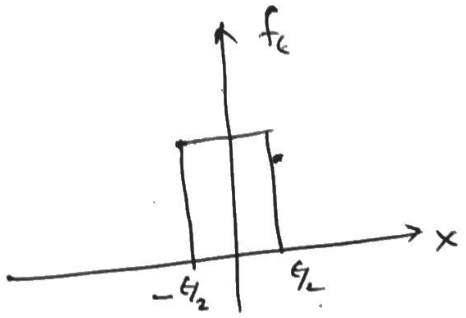
Laplace Equation :-

The dirac delta :-

$$f_\epsilon(x) = \begin{cases} 1/\epsilon , & |x| < \epsilon/2 \\ 0, & \text{else.} \end{cases}$$



$$\int_{-\infty}^{\infty} f_\epsilon(x)\, dx = 1$$

We define the dirac delta $\delta(x)$ as:

$$\delta(x) = \lim_{\epsilon \to 0} f_\epsilon(x).$$

Dirac delta is not a function, but a generalized fn or distribution. It has the following properties:

1) $$\delta(x) = \begin{cases} 0 , & x \neq 0 \\ \infty , & x = 0 \end{cases}$$

2) $$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

3) $$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0).$$

At $x = a$: $\delta(x-a) \Rightarrow$

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a. \end{cases}$$

Laplace eqn :-

Stationary heat or wave eqn:

$$u_{xx} = 0 \quad [\text{One -dimension}]$$

$$u_{xx} + u_{yy} = 0 \quad [\text{2-dim}]$$

A solution of the Laplace eqn is called a harmonic fn.

Non-homogeneous version :-

$$-(u_{xx} + u_{yy}) = f(x, y)$$

or $-\Delta u = f$ is called the

Poisson eqn.

1. Arise in Electrostatics: $\nabla \times E = 0 \Rightarrow E = -\nabla \phi$ [Maxwell Eqn]

$\nabla \cdot E = K \Rightarrow \Delta \phi = -K$.

2. Complex analysis: $f(z) = u(x,y) + i v(x,y)$ is analytic

$\Rightarrow u, v$ satisfies Cauchy - Riemann eqns.

$$u_x = v_x \,, \quad u_y = -v_x$$

Then, $\Delta u = 0$, $\Delta v = 0$.

Defn. u is sub-harmonic if $\Delta u \geq 0$, and super-harmonic if $\Delta u \leq 0$

Exm: $f(x,y) = xy$

$$f_{xx} + f_{yy} = 0.$$

H.W:- u, v are harmonic. Show that, uv is harmonic iff $\nabla u \cdot \nabla v = 0$.

H.W: Let, u is harmonic and u^2 is harmonic. Then prove that, u is a constant.

H.W: The Laplacian operator is the trace of the Hessian matrix.

In polar co-ordinates:

$$\Delta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

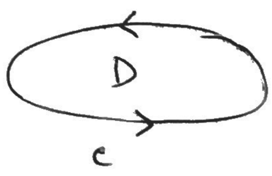
Green's Theorem:-

(Relation between double integral and line integral)

Let, C be a +vely oriented (D is on the left of c) simple, closed, piecewise smooth curve bounding the region D. If $P(x,y)$ & $Q(x,y)$ are cont. and have cont. p.ds in D, then,

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad \cdots (*)$$

$$\Rightarrow \int_{\partial D} (P\,dx + Q\,dy) = \iint_D (Q_x - P_y)\,dx\,dy$$



$$\Rightarrow \quad \partial_n u = \frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$$: directional derivative or normal derivative in the direction of $\vec{n}$.

If we put, $P = -V$, $Q = U$, then $(\ast)$ becomes:

$$\iint_D (U_x + V_y)\, dA = \int_{\partial D} (U\,dy - V\,dx)$$

$\vec{T} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$

$$= \int_{\partial D} (U, V)\cdot\left(\frac{dy}{ds}, -\frac{dx}{ds}\right) ds$$

$$= \int_{\partial D} (U, V)\cdot \vec{n}\, ds$$: $\hat{n}$ is the normal to the bdry ∂D.

So,

$$\iint_D (U_x + V_y)\, dA = \int_{\partial D} (U, V)\cdot \vec{n}\, ds \quad \cdots\cdots (\bigstar)$$

Now, let, $U = \psi \phi_x$, $V = \psi \phi_y$. then $(\bigstar)$ reduces to:

$$\iint_D \left(\psi(\phi_{xx} + \phi_{yy}) + \psi_x \phi_x + \psi_y \phi_y\right) dA = \int_{\partial D} \psi \nabla\phi \cdot \vec{n}\, ds$$

$$= \int_{\partial D} \psi \frac{\partial \phi}{\partial n}\, ds. \quad \cdots (I)$$

Ill.y, interchanging ψ & ϕ we get

$$\iint_D [\phi(\psi_{xx} + \psi_{yy}) + \psi_x \phi_x + \psi_y \phi_y]\, dA = \int_{\partial D} \phi \frac{\partial \psi}{\partial n}\, ds \quad \cdots (II)$$

(I) - (II) $\Rightarrow$

$$\iint_D (\psi \Delta\phi - \phi \Delta\psi)\, dA = \int_{\partial D} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) ds. \quad \cdots (A)$$

This is called the Green's Identity.

In $\mathbb{R}^n$:

$$\iint_D \psi \Delta\phi \, dA = - \iint_D \sum_{i=1}^{n} \psi_{x_i} \phi_{x_i} \, dA + \int_{\partial D} \psi \frac{\partial \phi}{\partial n} \, ds. \quad \cdots (B)$$

If $\psi = 1$., then, (A) $\Rightarrow$

$$\iint_D \Delta\phi = \int_{\partial D} \frac{\partial \phi}{\partial n} \, ds \quad \cdots (C).$$

If $\phi = \psi$. then (B) $\Rightarrow$

$$\iint_D \phi \Delta\phi \, dA = - \iint_D \sum_{i=1}^{n} \phi_{x_i}^2 \, dA + \int_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds \quad \cdots (D)$$

This is called the Energy Identity.

Boundary Value Problems :-

1. Dirichlet problem :

$$\left. \begin{aligned} -\Delta u &= f \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D. \end{aligned} \right\} \quad \cdots (III)$$

$\Rightarrow$ The soln of (III), if exists, is unique.

Let, u_1, u_2 are two soln. Then $v = u_1 - u_2$

Solves :

$$\Delta v = 0 \quad \text{in } D$$
$$v = 0 \quad \text{on } \partial D.$$

Now, by Energy Identity,

$$\iint_D \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 dA = 0$$

$$\Rightarrow \frac{\partial v}{\partial x_i} = 0 \quad \forall i \text{ in } D. \Rightarrow v = \text{constant in } D.$$

But, $v = 0$ on ∂D. and $v \in C^2(\bar{D})$, so,

$v = 0$ on D.

$\Rightarrow u_1 = u_2$ (Unique)

2. Neumann Problem :-

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } D \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D. \end{aligned}\right\} \cdots (\text{IV})$$

$\Rightarrow$ The soln of the Neumann problem is unique upto an addition of a constant.

Let, u_1, u_2 solve (IV), then $v = u_1 - u_2$ solves

$$\left.\begin{aligned} \Delta v &= 0 \quad \text{in } D \\ \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial D \end{aligned}\right\}$$

Now, by energy identity (D).

$$\frac{\partial v}{\partial x_i} = 0 \quad \forall i \quad \text{in } D.$$

So, $v =$ Const. in D.

Hence, $u_1 - u_2 =$ Const. in D.

Compatibility Condn :-

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D \\ \frac{\partial u}{\partial n} &= f \quad \text{on } \partial D. \end{aligned}$$

Then, (C) $\Rightarrow$

$$\int_{\partial D} f(s)\, ds = 0. \quad (\text{Neccessary condn for soln})$$

H.W. Prove that the solⁿ of the Robin's problem for the Laplace eqⁿ is unique, when $\alpha > 0$ is a const.

$$\Delta u = 0 \quad \text{in } D$$

$$\frac{\partial u}{\partial n} + \alpha u = f \quad \text{on } \partial D.$$

Dirichlet problem on a Rectangle :—

$$(1) \cdots \begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < l, \quad 0 < y < L \\ u(0,y) = u(l,y) = 0 \\ u(x,0) = \phi(x), \quad u(x,L) = 0 \end{cases}$$

let, the solⁿ of (1) is of the form:

$$u(x,y) = F(x)\,G(y). \quad \text{Then,}$$

$$\frac{F''}{F} + \frac{G''}{G} = 0$$

$$\therefore \frac{F''}{F} = -\frac{G''}{G} = \lambda, \text{ say}$$

$$F'' - \lambda F = 0 \quad \& \quad G'' + \lambda G = 0$$

$$u(0,y) = u(l,y) = 0 \Rightarrow F(0) = F(l) = 0$$

$$\& \quad G(L) = 0.$$

For non-trivial solⁿ, $\lambda < 0$, $\lambda = -\mu^2$, $\mu > 0$

$$\therefore F'' + \mu^2 F = 0, \quad F(0) = F(l) = 0$$

$$\Rightarrow F(x) = A \sin(\mu x) + B\cos(\mu x)$$

$$F(0) = 0 \Rightarrow B = 0$$

$$F(l) = 0 \Rightarrow \sin(\mu l) = 0 \Rightarrow \mu_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

$$\therefore F_n(x) = A_n \sin\left(\frac{n\pi x}{l}\right).$$

$$G'' - \mu^2 G = 0$$

$$\Rightarrow G(y) = C \sinh(\mu(L-y)) + D\cosh(\mu y)$$

$$G(L) = 0 \quad \Rightarrow \quad D = 0$$

$$\therefore \quad G(y) = C\sinh(\mu \, (L-y))$$

i.e. $$G_n(y) = C_n \sinh\left(\frac{n\pi}{l}(L-y)\right)$$

$$\therefore \quad u(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) \sinh\left(\frac{n\pi}{l}(L-y)\right)$$

with $$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi L}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \quad D_n = \frac{2}{l\sinh\left(\frac{n\pi L}{l}\right)} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\therefore \quad u(x,y) = \sum_{n=1}^{\infty} \left(\frac{2}{l\sinh\left(\frac{n\pi L}{l}\right)} \int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right) dx \right) \sin\left(\frac{n\pi x}{l}\right) \sinh\left(\frac{n\pi(L-y)}{l}\right)$$

(II) -- $$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < l, \; 0 < y < L \\ u(x,0) = \phi(x), \; u(x,L) = 0 \\ u_x(0,y) = 0 = u_x(l,y) \end{cases}$$

Two sides insulated. Let, $u(x,y) = F(x)G(y)$.

$$\frac{F''}{F} + \frac{G''}{G} = 0$$

$$\therefore \quad F'' - \lambda F = 0, \qquad G'' + \lambda G = 0$$

$$F'(0) = F'(l) = 0 \qquad G(L) = 0$$

For non-trivial soln, $\lambda = -\mu^2, \; \mu \geq 0$.

$$\therefore \quad F(x) = A\sin(\mu x) + B\cos(\mu x)$$

$$F'(0) = 0 \Rightarrow A = 0$$

$$\therefore \quad F(x) = B\cos(\mu x), \quad F'(l) = 0 \Rightarrow \mu_n = \frac{n\pi}{l}$$

$n = 0, 1, 2, \ldots$

$$F_n(x) = B_n \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \dots$$

$$G'' - \mu^2 G = 0, \quad G(L) = 0$$

$$G(y) = \quad c \sinh(\mu(y-L))$$

$$\therefore G_n(y) = C_n \sinh\left(\frac{n\pi(y-L)}{l}\right), \quad n = 0, 1, 2, \dots$$

$$\therefore u(x,y) = \sum_{n=0}^{\infty} D_n \sinh\left(\frac{n\pi(y-L)}{l}\right) \cos\left(\frac{n\pi x}{l}\right). \quad \dots (*)$$

Now, $u(x,0) = \phi(x) \Rightarrow$

$$\phi(x) = -\sum_{n=0}^{\infty} D_n \sinh\left(\frac{n\pi L}{l}\right) \cos\left(\frac{n\pi x}{l}\right).$$

with $$D_n = -\frac{2}{l \sinh\left(\frac{n\pi L}{l}\right)} \int_0^l \phi(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

H.W

$$\begin{cases} \Delta u = 0, & 0 < x < l, \quad 0 < y < L \\ u(x,0) = 0 = u(x,L) \\ u(0,y) = g(y), \quad u_x(l,y) = h(y) \end{cases}$$

Maximum Principle :- (Maximum - Minimum principle)

Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic fn in a bdd domain Ω. Then,

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u, \qquad \Gamma = \partial\Omega.$$

$$\min_{\bar{\Omega}} u = \min_{\Gamma} u.$$

Proof :-

Since $\partial\Omega \subseteq \bar{\Omega}$, $\max_{\Gamma} u \leq \max_{\bar{\Omega}} u$.

Sufficient to prove, $\max_{\bar{\Omega}} u \leq \max_{\Gamma} u$.

Consider the fn, $v(x) = u(x) + \epsilon|x|^2$, $x \in \bar{\Omega}$.

For $x \in \Omega$, $\Delta v = \Delta u + 2n\epsilon$

> 0

$|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$

$\Delta |x|^2 = \nabla \cdot (\nabla |x|^2)$

$= 2n$

Now, if v attains local max, at pt $x \in \Omega$, then,

$v_{x_i x_i} \leq 0$, $i = 1, 2, \ldots, n$

i.e. $\Delta v \leq 0$.

So, v does not attain maximum in Ω.

But, v has to have a maximum in $\bar{\Omega}$, so it is attained at $x^* \in \partial\Omega$. So, for $x \in \bar{\Omega}$

$$u(x) \leq v(x) \leq v(x^*) = u(x^*) + \epsilon|x^*|^2$$

$$\leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} |x|^2$$

So, $u(x) \leq \max_{\partial\Omega} u(x)$, $\forall x \in \bar{\Omega}$.

So, $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$.

Hence the result.

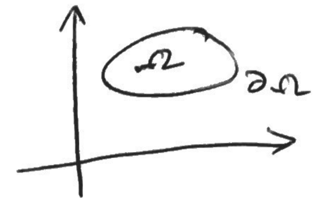
The minimum principle holds by applying max. principle to $w = -u$.

Alternating Proof :-

Let, $M = \max_{\partial\Omega} u$

Suppose $\max_{\bar{\Omega}} u$ is not attained in $\partial\Omega$.

Then, it must be attained inside Ω, at $P_0(x_0, y_0)$.

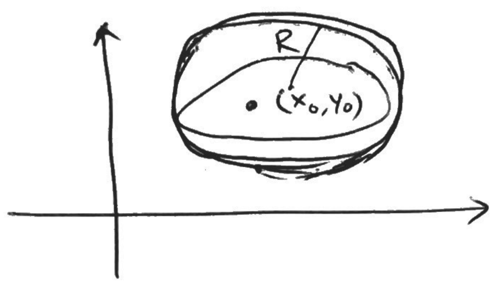


Let, $M_0 = u(x_0, y_0) = \max_{\Omega} u = \max_{\bar{\Omega}} u$

Then, $M \leq M_0$.

Consider $v(x,y) = u(x,y) + \dfrac{M_0 - M}{4R^2}\left[(x-x_0)^2 + (y-y_0)^2\right]$

where, $P(x,y) \in \Omega$ and R is the radius of a circle, containing Ω.



Note that, $v(x_0, y_0) = u(x_0, y_0) = M_0$ &

$\sqrt{(x-x_0)^2 + (y-y_0)^2} = d(P, P_0) < 2R$ for $(x,y) \in \bar{\Omega}$.

So, $v(x,y) < M + \dfrac{(M_0 - M)}{4R^2} 4R^2 = M_0$ on $\partial\Omega$.

Therefore, $v(x,y)$ must attain its max inside Ω. Now,

$$\Delta v = \Delta u + \frac{M_0 - M}{R^2} = \frac{M_0 - M}{R^2} > 0 \text{ in } \Omega.$$

But, for maximum inside Ω,

$v_{xx} \leq 0$, $v_{yy} \leq 0$ i.e. $\Delta v \leq 0$,

a contradiction.

So, the maximum of u must be attained on $\partial\Omega$.

✱ **Thm 15**

The solⁿ of the Dirichlet problem, it it exists, is unique.

Proof Let, u_1 & u_2 be two solⁿs of the Dirichlet problem:

$$\Delta u = f \text{ in } \Omega, \quad f \in C(\bar{\Omega})$$
$$u = \phi \text{ on } \partial\Omega, \quad \phi \in C(\partial\Omega)$$

Let, $v = u_1 - u_2$. Then, v is also harmonic and $v = 0$ on $\partial\Omega$. The max-min principle gives:

$$v = 0 \text{ in } \Omega.$$

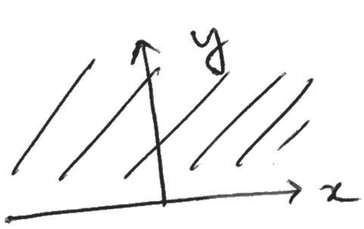
$$\text{i.e. } u_1 = u_2 \text{ in } \Omega.$$

This gives the uniqueness.

Remark :- For uniqueness, in the above thm, Ω needs to be bounded.

Counter Exm :-

$$\Delta u = 0, \quad x \in \mathbb{R}, \ 0 < y < \infty$$
$$u(x,0) = 0, \quad x \in \mathbb{R}.$$



$u_1(x,y) = xy$ & $u_2(x,y) = 0$ are two solⁿ of the Dirichlet problem.

Stability :-

Let, Ω be a bdd domain &

$u_i \in C^2(\Omega) \cap C(\bar{\Omega})$ solve

$$\Delta u_i = f \quad \text{in } \Omega,$$
$$u_i = g_i \quad \text{on } \partial\Omega, \quad \text{where}$$

$g_i \in C(\partial\Omega)$, $i = 1, 2$. Then, u_1 & u_2 satisfy

$$\max_{\bar{\Omega}} |u_1 - u_2| \leq \max_{\partial\Omega} |g_1 - g_2| .$$

Proof :- Define $\omega = u_1 - u_2$. Then ω satisfies:

$$\Delta \omega = 0 \quad \text{in } \Omega$$
$$\& \ \omega = g_1 - g_2 \quad \text{on } \partial\Omega .$$

By Max - Min principle,

$$\min_{\partial\Omega} (g_1 - g_2) \leq \omega(x,y) \leq \max_{\partial\Omega} (g_1 - g_2),$$

$\forall (x,y) \in \Omega$.

This results the proof.

Strong Maximum Principle :-

Let, $\Omega \subseteq \mathbb{R}^2$ be a connected domain (need not be bounded) and $u : \Omega \to \mathbb{R}$ be a harmonic fn. If u attains its maximum in Ω, then u is constant.

Proof :- See Evans.

Dirichlet Problem in a Disk :-

Consider

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } x^2+y^2 < a^2 \\ u(x,y) = \bar{g}(x,y) & \text{on } x^2+y^2 = a^2 \end{cases}$$

In polar co-ordinates, the eqn is:

$$U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad \text{for } 0 \le r < a, \; 0 < \theta \le 2\pi$$

with BC.: $U(a,\theta) = g(\theta)$, $0 \le \theta \le 2\pi$

where, $u(x,y) = U(r,\theta)$, with $x = r\cos\theta$, $y = r\sin\theta$.

and $\bar{g}(a\cos\theta, a\sin\theta) = g(\theta)$.

Seperation of variables :-

Assume the soln as:

$$U(r,\theta) = F(r)\,G(\theta). \quad \text{So.}$$

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0$$

$$\Rightarrow \quad r^2\frac{F''}{F} + r\frac{F'}{F} = -\frac{G''}{G} = \lambda, \text{ say}.$$

$$\therefore \quad r^2F'' + rF' - \lambda F = 0 \quad \& \quad G'' + \lambda G = 0.$$

Both U and g are 2π-periodic w.r.to θ.

So, G is 2π-periodic. i.e.

$$G(\theta) = G(2\pi + \theta).$$

$\lambda = 0$: $\quad G'' = 0 \Rightarrow G = A + B\theta$

Now, $G(\theta) = G(2\pi+\theta)$

$\Rightarrow A + B\theta = A + B\theta + 2\pi B$

$\Rightarrow B = 0.$

$\therefore \quad \lambda = 0, \quad G(\theta) = A \quad \&$

$$r^2 F'' + r F' = 0$$

$$\Rightarrow \int \frac{dF'}{F'} + \int \frac{dr}{r} = c$$

$$\therefore \ln(rF') = \ln c'$$

$$\Rightarrow \frac{dF}{dr} = \frac{c'}{r}$$

$$F(r) = c \ln(r) + D$$

$\therefore \lambda = 0 : \quad u(r,\theta) = c' \ln(r) + D'$ is a Soln.

$\underline{\lambda < 0}$: $\quad \lambda = -\mu^2. \qquad G'' - \mu^2 G = 0$

$$\therefore G(\theta) = A_1 e^{\mu\theta} + B_1 e^{-\mu\theta}$$

If A_1 or $B_1 \neq 0$, $|G(\theta)| \to \infty$ as $\theta \to \infty$, which contradicts periodicity.

$\underline{\lambda > 0}$: $\quad \lambda = \mu^2.$

$$G(\theta) = A \cos(\mu\theta) + B \sin(\mu\theta)$$

Now, $G(\theta) = G(2\pi + \theta)$ $\left[G(0) = G(2\pi) \;\&\; G'(0) = G'(2\pi) \right]$

$$\Rightarrow \mu = n, \quad \underline{n \in \mathbb{N}}$$

i.e. $\quad \lambda_n = n^2, \; n \in \mathbb{N}, \quad G_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$

Now, $\quad r^2 F'' + r F' - \mu^2 F = 0 \quad$ (Euler's eqn)

Put. $r = e^t \quad \Rightarrow [D(D-1) + D - \mu^2] F = 0$

$$D \equiv \frac{d}{dt}$$

$\Rightarrow \quad (D^2 - \mu^2) F = 0$

$\therefore \quad F(t) = c e^{\mu t} + E e^{-\mu t}$

$\therefore \quad F_n(r) = C_n r^n + E_n \frac{1}{r^n} \quad , \; n \in \mathbb{N}.$

So,

$$U_n(r,\theta) = \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)\left(C_n r^n + E_n \frac{1}{r^n}\right)$$

is a sol$\underline{n}$, for $n \in \mathbb{N}$

Also, $U_0(r,\theta) = A_0 + B_0 \ln(r)$ is a sol$\underline{n}$.

Now, since $\frac{1}{r^n} \to \infty$ or $\ln(r) \to -\infty$ as $r \to 0$, $E_n = 0 \; \forall n$ & $B_0 = 0$. So,

$$U(r,\theta) = \frac{K_0}{2} + \sum_{n=1}^{\infty} r^n \left(K_n \cos n\theta + l_n \sin n\theta\right) \quad \cdots (*)$$

is the sol$\underline{n}$. With the BC:

$U(a,\theta) = g(\theta)$ i.e,

$$g(\theta) = \frac{K_0}{2} + \sum_{n=1}^{\infty} a^n \left(K_n \cos n\theta + l_n \sin n\theta\right).$$

So,

$$\left.\begin{aligned} K_n &= \frac{1}{\pi a^n} \int_0^{2\pi} g(\theta) \cos n\theta \, d\theta \\ \& \; l_n &= \frac{1}{\pi a^n} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \end{aligned}\right\} \quad \cdots (**)$$

$(*)-(**)$ is the full sol$\underline{n}$ of the Dirichlet problem.

The series $(*)$ can be summed exactly!

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\frac{r}{a}\right)^n \int_0^{2\pi} g(\phi)\cos(\theta-\phi)n\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right] d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{r}{a}e^{-i(\theta-\phi)}\right)^n\right] d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\left[1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}}\right] d\phi$$

(Geometric Series $\left|\frac{r}{a}e^{i(\theta-\phi)}\right| < 1$)

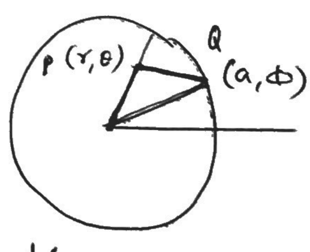
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{g(\phi)\,(a^2-r^2)}{a^2+r^2-2ar\cos(\theta-\phi)}\,d\phi.$$

$$\therefore\quad U(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi} \frac{g(\phi)}{a^2+r^2-2ar\cos(\theta-\phi)}\,d\phi \quad \text{--(I)}$$

This is called the Poisson's formula for a circle.

$\bar{x}$: (r,θ) in polar co-ordinates

$\bar{x}'$: (a,ϕ) on the circle.



then, $$u(\bar{x}) = \frac{a^2-|\bar{x}|^2}{2\pi a}\int_{|\bar{x}'|=a} \frac{u(\bar{x}')}{|\bar{x}-\bar{x}'|^2}\,ds, \quad \text{...(II)}$$

where $\underline{ds = a\,d\phi}$ arc length element.

(II) is the line-integral form of (I).

Mean Value Property :-

Let, u be a harmonic fn in a disk D, Continuous in its closure $\bar{D}$. Then, the value of u at the center of D equals the average of u on its circumference.

Proof :- (11) $\Rightarrow$

$$u(\bar{0}) = \frac{a^2}{2\pi a} \int_{|x'|=a} \frac{u(\bar{x}')}{a^2} \, ds$$

$$= \frac{1}{2\pi a} \int_{|x'|=a} u(\bar{x}') \, ds$$

This is the avg. of u on $|x'| = a$.

Remark :- If $g(\theta) = 1$, then, $(**) \Rightarrow$

$$k_n = l_n = 0 \quad , \quad n = 1, 2, \ldots$$

$$k_0 = 2.$$

So, $$U(r, \theta) = 1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2) \, d\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2}$$

$$\therefore \quad g(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \, g(\theta) \, d\phi$$

Non-homogeneous Problem :-

$$(*) \ldots \begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Write $u = v + w$ with

$$(**) \ldots \begin{cases} \Delta v = f & \& \quad \Delta w = 0 \quad \text{in } \Omega \\ & \quad w = -v + g \quad \text{on } \partial\Omega. \end{cases}$$

where v is a particular soln. of the poisson eqn.

$\Rightarrow$ If f is a polyn of degree n, we seek for soln in polyn of degree $(n+2)$.

Exm

$$\Delta u = -2, \quad 0 < x < a, \quad 0 < y < b$$

$$u(0,y) = 0 = u(a,y), \quad u(x,0) = 0 = u(x,b)$$

Let, $v(x,y) = A + Bx + Cy + Dx^2 + Ey^2 + Exy$

Substituting in $(**)$:

$$2D + 2F = -2$$

Let, $F = 0$, $D = -1$, The remaining co-eff are arbitrary.

Therefore, $v(x,y) = ax - x^2$, so that v reduces to 0 on $x=0$, $x=a$.

To find $\Delta w = 0, \quad 0 < x < a, \quad 0 < y < b$

$$w(0,y) = -v(0,y) = 0$$

$$w(a,y) = -v(a,y) = 0$$

$$w(x,0) = -v(x,0) = -(ax - x^2)$$

$$w(x,b) = -v(x,b) = -(ax - x^2)$$

By Separation of variables. we get

$$\omega(x,y) = \sum_{n=1}^{\infty}\left[a_n \cosh\left(\frac{n\pi y}{a}\right) + b_n \sinh\left(\frac{n\pi y}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right)$$

with

$$-(ax-x^2) = \omega(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)$$

&

$$-(ax-x^2) = \omega(x,b) = \sum_{n=1}^{\infty}\left[a_n \cosh\left(\frac{n\pi b}{a}\right) + b_n \sinh\left(\frac{n\pi b}{a}\right)\right]. \sin\left(\frac{n\pi x}{a}\right).$$

So, $$a_n = \frac{2}{a}\int_0^a (x^2-ax)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \begin{cases} 0, & n \text{ even} \\ -\frac{8a^2}{n^3\pi^3}, & n \text{ odd}. \end{cases}$$

$$a_n \cosh\left(\frac{n\pi b}{a}\right) + b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a (x^2-ax)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$\therefore \quad b_n = \frac{1-\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\, a_n .$$

So, $\underline{u = (ax-x^2) + \omega(x,y)}$

Fundamental Solution :-

If $|\bar{x}| = |\bar{y}| \Rightarrow u(\bar{x}) = u(\bar{y})$, then u is called a radial sol$\underline{n}$.

$$\Delta u = 0 \quad \cdots (*) \quad \text{in } \mathbb{R}^n - \{0\}.$$

Let, $r^2 = \sum_{i=1}^{n} x_i^2$ for $\bar{x} = (x_1, x_2, \cdots, x_n)$

Let, $\exists$ a radial sol$\underline{n}$ of $(*)$. i.e.

$$u(\bar{x}) = v(r) \quad \text{where, } r = |\bar{x}|$$

$$\frac{\partial u}{\partial x_i} = v'(r) \frac{\partial r}{\partial x_i}$$

$$= v'(r) \frac{x_i}{r}.$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{v'(r)}{r} + \frac{r v''(r) - v'(r)}{r^2} \frac{x_i^2}{r}$$

$$\therefore \Delta u = \sum v''(r) \left(\frac{x_i}{r}\right)^2 + \frac{v'(r)}{r} \sum \left(1 - \frac{x_i^2}{r^2}\right)$$

$$= v''(r) + \frac{(n-1)\, v'(r)}{r} = 0.$$

$$\Rightarrow \frac{v''}{v'} = \frac{(1-n)}{r}$$

$$\Rightarrow \ln v' = (1-n) \ln r + \ln c.$$

$$\Rightarrow v'(r) = c r^{1-n}$$

$$\therefore v(r) = \begin{cases} c \ln r + d, & n = 2 \\ \frac{c}{2-n} r^{2-n} + d, & n \neq 2. \end{cases}$$

For $n=1$, $u(x) = cx+d$ is a harmonic fn.

$$u(\bar{x}) = \begin{cases} c \ln|\bar{x}|, & n=2 \\ \dfrac{c}{2-n} \dfrac{1}{|\bar{x}|^{n-2}}, & n \geq 3. \end{cases} \quad \begin{matrix}(d=0) \\ \underline{(\text{Simplify})}\end{matrix}$$

As $|\bar{x}| \to 0$, $|u(\bar{x})| \to \infty$, so, $u(\bar{x})$ satisfies

$$\Delta u = \delta(\bar{x}).$$

We take c such that $\displaystyle\int_{S_r(0)} v'(r)\, ds = 1$. for every sphere

i.e. $$1 = \begin{cases} \dfrac{c}{r} 2\pi r, & n=2 \\ c\, r^{1-n} \cdot r^{n-1} \omega_n, & n \geq 3 \end{cases}$$

$$\therefore c = \begin{cases} \dfrac{1}{2\pi}, & n=2 \\ \dfrac{1}{\omega_n}, & n \geq 3 \end{cases}$$

where, ω_n is the area of n-dim unit sphere,

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})},$$

$$\Gamma(t) = \int_0^\infty e^{-\alpha} \alpha^{t-1}\, d\alpha, \quad \underline{t>0}.$$ is the Gamma function. or, $\omega_n = n\,\alpha(n)$, $\alpha(n)$ = volume of unit sphere.

For any fixed $\bar{x}_0 \in \mathbb{R}^n$, we define

$$\mathbb{K}(\bar{x}_0, \bar{x}) := \begin{cases} -\dfrac{1}{2\pi} \ln|\bar{x}-\bar{x}_0|, & n=2 \\ \dfrac{|\bar{x}-\bar{x}_0|^{2-n}}{\omega_n (n-2)}, & n \geq 3 \end{cases}$$

is the fundamental soln of Δ at $\bar{x}_0 \in \mathbb{R}^n$.

$$\Rightarrow \quad -\Delta \mathbb{K}(\bar{x}_0, \bar{x}) = \delta(\bar{x}-\bar{x}_0) \text{ in } \mathbb{R}^n \quad \left[\because \delta(\bar{x}-\bar{x}_0) = \begin{cases} \infty, & \bar{x}=\bar{x}_0 \\ 0, & \text{else} \end{cases}\right]$$

P.T.O

Thm 16 For any $f \in C_c^2(\mathbb{R}^n)$, $u = \mathbb{K} * f$ is a soln of the Poisson problem:

$$-\Delta u = f \text{ in } \mathbb{R}^n.$$

$\{x \in X \mid f(x) \neq 0\}$ [f vanishes outside a compact set]

Hint:

$$u(\bar{x}_0) = \int_{\mathbb{R}^n} \mathbb{K}(\bar{x}_0 - \bar{y}) f(\bar{y})\, d\bar{y}$$

$$-\Delta u(\bar{x}_0) = -\int_{\mathbb{R}^n} \Delta_x \mathbb{K}(\bar{x}_0 - \bar{y}) f(\bar{y})\, d\bar{y} \quad [\because \Delta_x \mathbb{K} = \Delta_y \mathbb{K}]$$

$$= -\int_{\mathbb{R}^n} \Delta_y \mathbb{K}(\bar{x}_0 - \bar{y}) f(\bar{y})\, d\bar{y} \quad [\because -\Delta_y \mathbb{K} = \delta(y - x_0)]$$

$$= \int_{\mathbb{R}^n} f(\bar{y}) \quad \delta(\bar{y} - \bar{x}_0)\, d\bar{y}$$

$$= f(\bar{x}_0). \quad \left[\because \int_{\mathbb{R}^n} \delta(x) f(x)\, dx = f(0)\right]$$

***For n=2:** $\mathbb{K}(\bar{z}, \bar{z}_0) = -\frac{1}{2\pi} \ln\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$ $\left[\begin{array}{l} \bar{z} = (x, y) \\ \bar{z}_0 = (x_0, y_0) \end{array}\right]$

$$= -\frac{1}{4\pi} \ln\left((x-x_0)^2 + (y-y_0)^2\right)$$

$$\frac{\partial^2 \mathbb{K}}{\partial x^2} = -\frac{1}{4\pi} \frac{2\left[-(x-x_0)^2 + (y-y_0)^2\right]}{\left[(x-x_0)^2 + (y-y_0)^2\right]^2}$$

$$\frac{\partial^2 \mathbb{K}}{\partial^2 y} = -\frac{1}{4\pi} \frac{2\left[(x-x_0)^2 - (y-y_0)^2\right]}{\left[(x-x_0)^2 + (y-y_0)^2\right]^2}$$

$\therefore \Delta \mathbb{K} = \delta(\bar{z} - \bar{z}_0)$, blows up when $\bar{z} = (x, y) = (x_0, y_0) = \bar{z}_0$.

Problems :-

1. Solve $\Delta u = 1$ in $\sqrt{x^2+y^2} < a$ with $u = 0$ on $x^2+y^2=a^2$.

$$\left(\text{Ans: } u(x,y) = \frac{1}{4}(x^2+y^2-a^2)\right)$$

$u = v + w$: $\Delta v = 1$, $\Delta w = 0$
$w = -v$ on $r = a$

2. Solve $\Delta u = 0$ in $x^2+y^2<a^2$ with $u = 1 + 3\sin\theta$ on $r = a$

$$\left(\text{Ans: } u(r,\theta) = 1 + \frac{3r}{a}\sin\theta\right)$$

3. Solve

$$\Delta u = 0, \quad 0 < x < \pi, \quad 0 < y < \pi$$
$$u(x,0) = u(x,\pi) = 0$$
$$u(0,y) = 0, \quad u(\pi, y) = \sin y$$

Is there a pt. (x,y) in $(0,\pi)\times(0,\pi)$ s.t. $u(x,y) = 0$?

4. Let, $\Omega \subseteq \mathbb{R}^2$ is bdd. $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be s.t. $\Delta u \geq 0$ in Ω. Prove that its max value is attained on the boundary. What can you say about the minimum value?

5. Let, u be a harmonic fn on the whole plane s.t. $u = 3\sin(2\theta) + 1$ on the circle $x^2+y^2=2$. Without finding the concrete form of the soln, find the value of u at the origin.

6. Solve $\Delta u = 0$, $x^2+y^2 > a^2$

$u(x,y) = h(x,y)$ on $x^2+y^2=a^2$

u is bounded as $x^2+y^2 \to \infty$.

7. Suppose u is harmonic in $r<2$ and $u = 3\sin(2\theta)+1$ for $r=2$. Find the max value of u in $r \leq 2$.

8. Prove that, $\ln(x^2+y^2)$ is harmonic.