6.8 ①

Indeterminate Forms & L'Hospital's Rule :-

$$F(x) = \frac{x-3}{x-1}, \quad x \neq 1.$$

Domain does not include 1. but can be analyzed limit value at 1.

$$\lim_{x \to 1} \frac{x-3}{x-1} = \infty.$$

What about $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\left(\frac{0}{0}\right)$.

1. $\lim_{x \to a} \frac{f(x)}{g(x)}$ $\left(\frac{0}{0}\right)$ as $f(x) \to 0$, $g(x) \to 0$ as $x \to a$.

2. Limit at Infinity : $\lim_{x \to \infty} \frac{f(x)}{g(x)}$. $\left(\frac{\infty}{\infty}\right)$

as $f(x) \to \infty$ & $g(x) \to +\infty$ as $x \to \infty$.

L'Hospital's Rule :- Let, f & g are differentiable and $g'(x) \neq 0$. near a and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or}$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty \quad \text{then,}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1.$$

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$$\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2x} = \lim_{x\to\infty} \frac{2^x\cdot(\ln 2)^2}{2} = \infty.$$

$\Rightarrow$ exponential fn. grows more rapidly than power fn.!

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$$\lim_{x\to\pi} \frac{\sin x}{1-\cos x} = \lim_{x\to\pi} \frac{\cos x}{\sin x} = -\infty \quad \text{(WRONG)}$$

as $\left(\frac{0}{2}\right)$ form!

L'Hospital's Rule does not apply!

Product Rule: $\lim_{x\to a} fg$ $\quad [\, 0\cdot\infty$ or $0\cdot(-\infty)$ form$\,]$

$$fg = \frac{f}{1/g} \text{ or } \frac{g}{1/f}.$$

Differences:- $\lim_{x\to a}\left(f(x) - g(x)\right)$ $\quad$ ($\infty - \infty$ or $-\infty + \infty$ form)

Convert it to $\frac{0}{0}$ or ∞/∞ form. algebraically

Power: $\lim_{x\to a} [f(x)]^{g(x)}$ $\quad$ (0^0, ∞^0, 1^∞ form)

$\lim_{x\to 0+} x^x.$

$$y = \lim_{x\to 0+} x^x$$

$$\ln y = \lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0+} x = 0$$

$\therefore\ y = e^0 = 1$

12 $\lim_{x\to 0} \frac{\sin 4x}{\tan 5x} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \frac{4}{5}$

16 ?

22. $\lim_{t\to 0} \frac{8^t - 5^t}{t} = \lim_{t\to 0} \frac{8^t \ln 8 - 5^t \ln 5}{1} = \ln 8 - \ln 5.$

20. $\lim_{x\to \infty} \frac{\ln\sqrt{x}}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{2x} = \lim_{x\to\infty} \frac{1}{4x^2} = 0.$

25. $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$

42. $\lim_{x\to\infty} \sqrt{x}\, e^{-x/2} \quad (\infty \cdot 0)$

$= \lim_{x\to\infty} \sqrt{x}/e^{x/2} \quad \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1/2\sqrt{x}}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\, e^{x/2}}$

$= 0.$

58. $y = \lim_{x\to\infty} \left(1 + a/x\right)^{bx}$

$\ln y = \lim_{x\to\infty} bx \ln\left(1 + a/x\right)$

$= \lim_{x\to\infty} \frac{\ln(1 + a/x)}{1/bx} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to\infty} \frac{\frac{1}{1+a/x}\cdot\left(-a/x^2\right)}{-1/bx^2} = \lim_{x\to\infty} \frac{ab}{1 + a/x} = ab$

$\therefore\ y = e^{ab}.$

71. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \lim\limits_{x\to\infty} \dfrac{e^x}{nx^{n-1}} = \lim\limits_{x\to\infty} \dfrac{e^x}{n(n-1)x^{n-2}}$

$$= \cdots$$

$$= \lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$

So, exponential approaches infinity faster than power of x.

72. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^p} = \lim\limits_{x\to\infty} \dfrac{1/x}{px^{p-1}} = \lim\limits_{x\to\infty} \dfrac{1}{px^p} = 0$

So, log approaches to ∞ more slowly than any power!

<u>Q.H.W.</u> $\lim\limits_{x\to 0+} \left(\dfrac{1}{\ln(1+x)} - \dfrac{1}{x} \right) \qquad (\infty - \infty)$

$$\# \qquad \lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x} \qquad (0^0)$$

$$y = \lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x}$$

$$\Rightarrow \ln y = \lim_{x\to\infty} \frac{\ln(1/x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1/x}\left(-\frac{1}{x^2}\right)}{1}$$

$$= \lim_{x\to\infty} \left(-\frac{1}{x}\right) = 0.$$

$$\therefore \; \underline{y = e^0 = 1}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \underline{1}.$$