Variation of Parameters :-

$$\frac{dy}{dx} + P(x)y = Q(x). \quad -- (*)$$

S1 : Consider the homogeneous eqn.

$$\frac{dy}{dx} + Py = 0$$

Let, $y = c_1 u$ is a solution.

S2: Let, the solution of $(*)$ is

$$y = uv. \quad \text{with } v \text{ is a function of } x \quad -- (1)$$

$$u'v + uv' + Puv = Q$$

$$\Rightarrow v' + \left(P + \frac{u'}{u}\right) v = Q/u \Rightarrow v' = \frac{Q}{u}.$$

$$\frac{(u' + Pu)}{u} = 0$$

Solve for v. and replace in (1).

$$\therefore v = \int \left(Q/u\right) dx + c.$$

Exm. $(x+4)\frac{dy}{dx} + 3y = 3.$

$$\Rightarrow \frac{dy}{dx} + \frac{3}{(x+4)} y = \frac{3}{(x+4)}.$$

S1. $\frac{dy}{dx} + \frac{3}{(x+4)} y = 0$

$$\Rightarrow \ln y + 3\ln(x+4) = A$$

$$\therefore y = c(x+4)^{-3}$$

S2: The required gen. solution be:

$$y = v(x)(x+4)^{-3}$$

This satisfies:

$$v' = \frac{3}{(x+4)} (x+4)^{3} = 3(x+4)^2$$

$$\therefore v(x) = (x+4)^3 + K.$$

So, $y(x) = K(x+4)^{-7} + 1$

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$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x) \quad \cdots (\cancel{*})$$

S1. Let, the general soln of
$y'' + Py' + Qy = 0$ is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

then we let the gen. soln of $(\cancel{*})$ be

$$y(x) = u_1 y_1 + u_2 y_2$$, u_1 & u_2 are x dependant.

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$\cancel{y''}$ Let, $u_1' y_1 + u_2' y_2 = 0 \quad \cdots$ ①

then $y'' = u_1 y_1'' + u_1' y_1' + u_2' y_2' + u_2 y_2''$

Since y_1 & y_2 are soln of homogenous problems we get,

$$u_1' y_1' + u_2' y_2' = R(x) \quad \cdots \text{(II)}$$

Solve ① & (II) to get u_1' & u_2' and so u_1 & u_2.

If W = wronskian of y_1 & y_2 = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

then

$$u_1 = A - \int \frac{y_2 R}{W} dx$$

$$u_2 = B + \int \frac{y_1 R}{W} dx.$$

Exm. $y'' + n^2 y = \sec(nx)$

$y_c = c_1 \sin(nx) + c_2 \cos(nx)$

Let, $y_p = u \sin(nx) + v \cos(nx)$

where u & v satisfy:

$$u' \sin(nx) + v' \cos(nx) = 0 \quad \cdots (1)$$

$$\& \; n u' \cos(nx) - n v' \sin(nx) = \sec(nx)$$

$$\therefore \quad u(x) = -\int \frac{y_2 R(x)}{W} dx$$

$$\& \; v(x) = \int \frac{y_1 R(x)}{W} dx$$

where, $W(x) = \begin{vmatrix} \sin(nx) & \cos(nx) \\ n\cos(nx) & -n\sin(nx) \end{vmatrix}$

$$= -n \neq 0$$

$$u(x) = \frac{1}{n} \int \cos(nx)\sec(nx)\,dx = \frac{x}{n}$$

$$v(x) = -\frac{1}{n} \int \tan(nx)\,dx = \frac{1}{n^2} \int \frac{dz}{z}$$

$\cos(nx) = z$

$-n\sin(nx)dx = dz$

$$= \frac{1}{n^2} \ln(\cos(nx))$$

$$\therefore \; y_p = \frac{x}{n} \sin(nx) + \frac{1}{n^2} \ln(\cos(nx)) \cos(nx).$$

Exm. $y'' + 4y = 4\tan(2x)$

$y_c = A \sin(2x) + B \cos(2x)$

$W(x) = \begin{vmatrix} \sin(2x) & \cos(2x) \\ 2\cos(2x) & -2\sin(2x) \end{vmatrix} = -2$

Let, $y_p = u \sin(2x) + v \cos(2x)$

$$\cancel{u(x)} \quad u(x) = -\int \frac{y_2 R(x)}{W} dx$$

$$v(x) = \int \frac{y_1 R(x)}{W} dx$$

$$u(x) = \frac{1}{2}\int \cos(2x) \cdot 4\tan(2x)\, dx$$

$$= 2\int \sin(2x)\, dx = -\cos(2x)$$

$$v(x) = -\frac{1}{2}\int \sin(2x) \cdot 4\tan(2x)\, dx$$

$$= -2\int \frac{\sin^2(2x)}{\cos(2x)} dx$$

$$= -2\int \sec(2x)\,dx + 2\int \cos(2x)\, dx$$

$$= -\ln\left(\sec(2x) + \tan(2x)\right) + \sin(2x)$$

$$\therefore y_p = \cancel{-\tfrac{1}{2}\sin(4x)} - \cos(2x)\ln\left(\sec(2x) + \tan(2x)\right)$$

<u>ExM</u>. $y'' - y = \frac{2}{1+e^x}$

$$y_c = Ae^x + Be^{-x}$$

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -1 - 1 = -2 \neq 0$$

Let, $y_p = ue^x + ve^{-x}$

$$u(x) = +\int \frac{\cancel{2}e^{-x}}{(1+e^x)} dx = \int \frac{dx}{e^x(1+e^x)}$$

$e^x = z$

$e^x dx = dz$

$$= \int \frac{dz}{z^2(1+z)}$$

$$= \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z}\right) dz$$

$$= -e^{-x} - x + \ln(1+e^x)$$

$$v(x) = -\int \frac{e^x}{1+e^x}\,dx = -\ln(1+e^x)$$

$$\therefore\ y_p = -1 - xe^x + e^x\ln(1+e^x) - e^{-x}\ln(1+e^x)$$

H.W. $\quad y'' - 2y' = e^x \sin x$

$$y_p = -\frac{1}{2}e^x \sin x.$$

Exm $\quad x^2 y'' + 3xy' + y = \frac{1}{(1-x)^2}$

$$x = e^z \quad : \quad z = \ln x$$

Consider, $(D'(D'-1) + 3D' + 1)\,y = 0$

$$\Rightarrow (D'^2 + 2D' + 1)\,y = 0$$

$$\Rightarrow y = Ae^{-z} + Bze^{-z}$$

$$= \frac{A}{x} + B(\ln x)\frac{1}{x}$$

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = \frac{1}{x^2(1-x)^2}$$

Let, $y_p = \frac{u}{x} + v\ln(x)\frac{1}{x}$

with $W(x) = \begin{vmatrix} 1/x & \ln(x)\frac{1}{x} \\ -1/x^2 & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$

$$= \frac{1}{x^3} \neq 0$$

$$u(x) = -\int \frac{\cancel{x^2}\ln(x)}{\cancel{x^2}(1-x)^2}\,dx = \ln x - \ln(1-x) - \frac{\ln x}{1-x}$$

$$v(x) = \int (1-x)^{-2}\,dx = \frac{1}{1-x}.$$

$$y_p = \frac{1}{x}\ln\left(\frac{x}{1-x}\right)$$

H.W. $y'' + y = \frac{1}{1+\sin x}$

H.W. $y'' + k^2 y = f(x)$, $y(0) = 0$, $y'(0) = 0$

$$y(x) = \frac{1}{K}\int_0^x f(t)\sin k(x-t)\,dt$$

Simultaneous ODE :-

$$f_1(D)x + f_2(D)y = T_1(t) \quad \cdots (I)$$

$$g_1(D)x + g_2(D)y = T_2(t) \quad \cdots (II)$$

with $D \equiv \frac{d}{dt}$. To solve for x & y.

Method 1: Elimination of one variable:

$\Rightarrow$ Use (I) & (II) to eliminate x or y to get ODE in x or y only.

Exm. $\frac{dx}{dt} = 7x - y$, $\frac{dy}{dt} = 2x + 5y$

Let, $D \equiv \frac{d}{dt}$ So,

$$(D-7)x + y = 0$$

$$(D-5)y - 2x = 0$$

$$\Delta = \begin{vmatrix} D-5 & 1 \\ -2 & D-7 \end{vmatrix}$$

$$= (D-5)(D-7) + 2$$

degree of $D = 2$

$\parallel$

Arbitrary constants

We eliminate x:

$$\cancel{(D-5)y + (D-7)}$$

$$+2y + (D-7)(D-5)y = 0$$

$$(D^2 - 12D + 37)y = 0$$

$m = 6 \pm i$

$$y = e^{6t}(A\cos t + B\sin t)$$

$$2x = (D-5)y = \cancel{6}e^{6t}(A\cos t + B\sin t) + e^{6t}(-A\sin t + B\cos t)$$

$$\cancel{5e^{6t}(}$$

$$\therefore x = e^{6t}\left(\frac{B-A}{2}\sin t + \frac{A+B}{2}\cos t\right)$$

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Method 2 :- Differentiation :

Find $\frac{dx}{dt}$ & x and substitute in another eqn..

$D^2y - 2Dx - 5Dy = 0$ .. (III)

$\cancel{Dx - 7x + y = 0}$ .. (IV)

Also, $x = \frac{1}{2}(Dy - 5y)$ .. (V)

From 2nd eqn.

Substitute x & Dx in (IV)

$D^2y - 12Dy + 37y$ ✓

Exm $Dx + 2y + x = e^t$, $Dy + 2x + y = 3e^t$

$(D+1)x + 2y = e^t$

$2x + (D+1)y = 3e^t$

$\Delta = \begin{vmatrix} D+1 & 2 \\ 2 & D+1 \end{vmatrix}$

$\deg(\Delta) = 2$

$[(D+1)^2 - 4]x = -4e^t$

$(D^2 + 2D - 3)x = -4e^t$

$x_c = Ae^t + Be^{-3t}$

$x_p = -\frac{1}{D^2+2D-3}4e^t = -4\frac{1}{(D-1)(D+3)}e^t$

$= -4 \frac{1}{4} \cdot \frac{1}{D-1} e^t$

$= -e^t \cdot \frac{1}{D} = -te^t.$

$x(t) = Ae^t + Be^{-t} - te^t$

$y = \frac{1}{2}e^t - \frac{1}{2}(D+1)x$

$= e^t - Ae^t + Be^{-3t} + te^t$

Exm. $Dx = -ky \qquad Dy = kx$

$$D^2 x = -kDy = -k^2 x$$

$$\Rightarrow (D^2 + k^2) x = 0$$

$$\therefore \quad x(t) = A\sin(kt) + B\cos(kt)$$

$$y(t) = -\frac{k}{k}\left[A\cos(kt) - B\sin(kt)\right]$$

$$y = B\sin(kt) - A\cos(kt)$$

$$x^2 + y^2 = A^2 + B^2$$

Exm. $D^2 x + m^2 y = 0 \qquad D^2 y - m^2 x = 0$

$$(D^4 + m^4)\, x = 0$$

AE: $\alpha^4 + m^4 = 0$

$$\Rightarrow \alpha^4 + m^4 + 2\alpha^2 m^2 - 2\alpha^2 m^2 = 0$$

$$\Rightarrow (\alpha^2 + m^2 - \sqrt{2}\,\alpha m)(\alpha^2 + m^2 + \sqrt{2}\,\alpha m) = 0$$

$$\alpha = \pm m/\sqrt{2} \pm i\, m/\sqrt{2} = \frac{m}{\sqrt{2}}(\pm 1 \pm i)$$

$$x(t) = e^{m/\sqrt{2}\,t}\left(A_1\cos\left(\frac{mt}{\sqrt{2}}\right) + A_2\sin\left(\frac{mt}{\sqrt{2}}\right)\right) + e^{-m/\sqrt{2}\,t}\left(B_1\cos\left(\frac{mt}{\sqrt{2}}\right) + B_2\sin\left(\frac{mt}{\sqrt{2}}\right)\right)$$

$$y(t) = -\frac{1}{m^2} D^2 x \ldots$$

$$= e^{-mt/\sqrt{2}}\left(B_2\cos\left(\frac{mt}{\sqrt{2}}\right) - B_1\sin\left(\frac{mt}{\sqrt{2}}\right)\right) + e^{mt/\sqrt{2}}\left(A_1\sin\left(\frac{mt}{\sqrt{2}}\right) - A_2\cos\left(\frac{mt}{\sqrt{2}}\right)\right)$$

non-linear Part-3 :-

$$M\,dx + N\,dy = 0$$

$\Rightarrow$ If $M(x,y)$ & $N(x,y)$ are homogeneous functions of deg. n & $Mx+Ny \neq 0$, then $I.F = \frac{1}{Mx+Ny}$.

$\Rightarrow$ If $Mx+Ny=0$, then $1/xy$ or $1/x^2$ or $1/y^2$ is an I.F.

Exm. $(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \quad \frac{\partial N}{\partial x} = 4xy - 3x^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x^2 - 8xy$$

$$\frac{1}{M}\left(- \right) = \frac{4}{y} \quad \text{~~Not a fn of~~} = g(y)$$

$$\therefore I.F = e^{-\int 4/y\,dy} = 1/y^4. \checkmark$$

Also, $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$

$$= x^2y^2 \neq 0$$

$$I.F = \frac{1}{x^2y^2}$$

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx + \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 0$$

$$\Rightarrow -\frac{2}{x}dx + \frac{3}{y}dy + \frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow -2\ln x + 3\ln y + \frac{x}{y} = K.$$

$$\Rightarrow \ln\left(\frac{y^3}{x^2}\right) = K - x/y$$

$$\therefore \underline{y^3 = x^2 A e^{-x/y}}$$

Applications :-

Exm. Radium is known to decay at a rate proportional to the amount present. If the half life of radium is 1600 Yrs, what percentage of radium in a given sample after 800 Yrs? Determine the number of Ys after which only one-tenth of the original amount of radium would remain?

Soln $x(t)$: amount present at time t.

$$\frac{dx}{dt} = kx \qquad k : \text{-ve constant.}$$

$$\therefore \ln x = kt + c \implies x(t) = Ae^{kt}$$

Let, at $t=0$, amount is : x_0 So,

$$A = x_0 \quad \text{i.e} \quad x(t) = x_0 e^{kt}$$

Half life is 1600 Yr. : $x = x_0/2$ at $t = 1600$

$$\therefore \frac{x_0}{2} = x_0 e^{k.1600}$$

$$\implies k = -\frac{1}{1600}\ln(2).$$

Let, $x = x'$ at $t = 800$ So

$$x' = x_0 e^{800k} = x_0 e^{-\frac{1}{2}\ln 2}$$

$$= \frac{x_0}{\sqrt{2}} = 0.707\, x_0$$

So, the percentage remaining: $\frac{x'}{x_0} \times 100$

$$= 70.7\%.$$

Let, $t = t_1$ when $x = \frac{x_0}{10}$

$$\therefore \frac{x_0}{10} = x_0 e^{kt_1}$$

$$\implies \frac{1}{1600}\ln 2 . t_1 = \ln(10)$$

$$\therefore t_1 = \frac{1600\ln(10)}{\ln(2)} = 5317 \text{ Yr.}$$

Exm. (Newton's law of Cooling)

According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temp. of substance and that of the air. If the temp of the air is 290K and the substance cools from 370K to 330K in 10 min, find when the temp will be 295K.)

Soln :- Let, T be the temp. of the substance at time t. Then,

$$\frac{dT}{dt} = -\lambda(T-290)$$

$$\Rightarrow \int_{370}^{330} \frac{dT}{T-290} = \int_{0}^{10} -\lambda \, dt$$

$$\cancel{\Rightarrow \ln(T-290) = -\lambda t + c}$$

$$\Rightarrow \ln(T-290)\Big]_{370}^{330} = -\lambda t\Big]_{0}^{10}$$

$$\Rightarrow \ln\left(\frac{40}{80}\right) = -10\lambda$$

$$\therefore \lambda = \frac{1}{10}\ln 2$$

Let, at $t = t_1$, the temp becomes 295K.

So,
$$\int_{370}^{295} \frac{dT}{T-290} = -\frac{1}{10}\ln 2 \int_{0}^{t_1} dt$$

$$\Rightarrow \ln\left(\frac{5}{80}\right) = -\frac{1}{10}\ln 2 \,.\, t_1$$

$$\therefore t_1 = 40 \text{ min}$$