Homogeneous Linear Equations or Cauchy-Euler Equations :-

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = R(x)$$

is known as Cauchy-Euler equations.

To solve, let, $x = e^z$

$$\Rightarrow z = \ln x \quad \Rightarrow \frac{dz}{dx} = \frac{1}{x}.$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$x \frac{d}{dx} \equiv \frac{d}{dz} \quad \Rightarrow xD \equiv D'$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\therefore x^2 D \equiv D'^2 - D' \equiv D'(D'-1).$$

||ly, $\underline{x^n D^n y \equiv D'(D'-1) \cdots (D'-n+1) y}$

Exm. $x^2 y'' - 3xy' + 4y = 0$

Let, $x = e^z$. $\Rightarrow xD \equiv D'$

The equation will be converted as:

$$[D'(D'-1) - 3D' + 4]y = 0$$

$$\Rightarrow (D'^2 - 4D' + 4)\, y = 0 \qquad y = e^{\alpha z}$$

Auxiliary eqn is: $\alpha^2 - 4\alpha + 4 = 0 \Rightarrow \alpha = 2, 2$

$$y = (A + Bz) e^{2z} = (A + B \ln x) x^2.$$

H.W $(x^3 D^3 + 3x^2 D^2 - 2xD + 2)\, y = 0 \qquad D \equiv \frac{d}{dx}. \qquad \underline{\alpha = 1, 1, -2}$

Operator Method :—

Linear ODE :

$f(D)\, y \; \cancel{L(y)} = R(x) \quad \ldots (1)$

If y_c is a solution of $L(u) = 0$ and y_p is a particular solution of $L(u) = R'$, then the general solution of (1) has the form:

$$y = y_c + y_p .$$

y_c is called a complementary function, it contains arbitrary constants

y_p is called a particular solution, it does not contain any constant.

Defn: $\dfrac{1}{f(D)} R(x)$ is the function $\phi(x)$ such that

$$f(D)\, \phi(x) = R(x) .$$

$\dfrac{1}{f(D)}$ is the inverse of the operator $f(D)$.

Exm. $\dfrac{1}{D^2 + 3D} (2+6x) = x^2$

as $(D^2+3D)x^2 = 2+6x .$

Methods to find y_p :—

R1. $\dfrac{1}{D-\alpha} R(x) = e^{\alpha x} \int R e^{-\alpha x} \, dx$

R2. $\dfrac{1}{(D-\alpha)^n} e^{\alpha x} = \dfrac{x^n}{n!} e^{\alpha x}$.

Ex4. $(D^2+k^2)y = \cot(kx)$.

$y = y_c + y_p$

where $y_c = A\sin(kx) + B\cos(kx)$

and $y_p = \dfrac{1}{D^2+k^2}\cot(kx)$

$$= \frac{1}{(D+ik)(D-ik)}\cot(kx).$$

$$= \frac{1}{2ik}\left[\frac{1}{D-ik} - \frac{1}{D+ik}\right]\cot(kx)$$

$$\frac{1}{D-ik}\cot(kx) = e^{ikx}\int e^{-ikx}\cot(kx)\,dx$$

$$= e^{ikx}\int\left[\frac{\cos^2(kx)}{\sin(kx)} - i\cos(kx)\right]dx$$

$$= e^{ikx}\int\left[\operatorname{cosec}(kx) - \sin(kx) - i\cos(kx)\right]dx$$

$$= e^{ikx}\left[\frac{1}{k}\ln\left|\tan\frac{kx}{2}\right| + \frac{1}{k}\cos(kx) - \frac{i}{k}\sin(kx)\right]$$

$$= \frac{1}{k}\left(e^{ikx}\ln\left(\tan\frac{kx}{2}\right) + 1\right)$$

$$\int \operatorname{cosec}(kx)\,dx = \frac{1}{2}\int\frac{1}{\sin(kx/2)\cos(kx/2)}\,dx = \frac{1}{2}\int\frac{\sec^2(kx/2)}{\tan(kx/2)}\,dx = \frac{1}{k}\ln\left|\tan\frac{kx}{2}\right| + A$$

Replace i by $-i$ and then we get

$$y_p = \frac{1}{2ik}\left[\frac{1}{k}\left(e^{ikx}\ln\left(\tan\frac{kx}{2}\right)+1\right) - \frac{1}{k}\left(e^{-ikx}\ln\left(\tan\frac{kx}{2}\right)+1\right)\right]$$

$$= \frac{\ln\left(\tan\frac{kx}{2}\right)}{k^2}\sin(kx)$$

H.W $(D^2+k^2)y = \sec(kx)$. $y_p = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx)$.
$\ln(\cos kx)$

E.M. $y'' + y = \sec^2 x$.

$\Rightarrow$ $y = A\cos x + B\sin x + y_p$

with $y_p = \frac{1}{D^2+1}\sec^2 x$.

$= \frac{1}{(D-i)(D+i)}\sec^2 x$

$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec^2 x$

$y'' + y = 0$
$m^2 + 1 = 0$
$m = \pm i$
$y_c = Ae^{ix} + Be^{-ix}$
$= A_1\cos x + B_1\sin x$

$\frac{1}{D-i}\sec^2 x = e^{ix}\int e^{-ix}\sec^2 x\,dx$

$= e^{ix}\int \frac{\cos x - i\sin x}{\cos^2 x}\,dx$

$= e^{ix}\int(\sec x - i\sec x\tan x)\,dx$

$= e^{ix}(\ln(\sec x + \tan x) - i\sec x)$

$\therefore y_p = \frac{1}{2i}\left[e^{ix}(\ln(\sec x + \tan x) - i\sec x) - e^{-ix}(\ln(\sec x + \tan x) + i\sec x)\right]$

$= \ln(\sec x + \tan x)\sin x - \sec x\cos x$

$= \sin x\ln(\sec x + \tan x) - 1$.

H.W

$y'' + y = \csc^2 x$.

R3. $$\frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} \cdot e^{\alpha x} \quad \text{if } f(\alpha) \neq 0.$$

\# $$\frac{1}{f(D)} 1 = \frac{1}{f(D)} e^{0x} = \frac{1}{f(0)} \quad \text{if } f(0) \neq 0$$

\# $$\frac{1}{D^n} 1 = \frac{1}{(D-0)^n} e^{0x} = \frac{x^n}{n!} e^{0.x} = \frac{x^n}{n!}$$

R4. If $f(D) = g(D^2)$ and $g(-a^2) \neq 0$, then

$$\frac{1}{f(D)} \sin(ax) = \frac{1}{g(-a^2)} \sin(ax)$$

$$\& \quad \frac{1}{f(D)} \cos(ax) = \frac{1}{g(-a^2)} \cos(ax)$$

Exm. $(D^2+1) y = \cos(2x)$

$$y_c = c_1 \sin x + c_2 \cos x$$

$$y_p = \frac{1}{D^2+1} \cos(2x) = \frac{1}{-4+1} \cos(2x)$$

$$= -\frac{1}{3} \cos(2x)$$

R5. $$\frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D+a)} V(x).$$

Exm. $(D^2+4) y = \sin(2x)$

$$y_c = c_1 \sin(2x) + c_2 \cos(2x)$$

$$y_p = \frac{1}{D^2+4} \sin(2x)$$

$$= \text{imaginary part of } \frac{1}{D^2+4} e^{i2x}.$$

$$= \operatorname{Im}\left[e^{2ix} \frac{1}{(D+2i)^2+4} 1 \right]$$

$$= \operatorname{Im}\left[e^{2ix} \frac{1}{D(D+4i)} e^{0x} \right]$$

$$= \operatorname{Im}\left[e^{2ix} \frac{1}{4i} \frac{1}{D} \cdot 1 \right] = \operatorname{Im}\left[\frac{x}{4i} e^{2ix} \right]$$

$$= -\frac{x}{4}\cos(2x).$$

Ill.y $\frac{1}{D^2+a^2}\cos(ax) = Re\left[\frac{1}{D^2+a^2}e^{iax}\right]$

$$= \frac{x}{2a}\sin(ax).$$

Exm. $(D^2-3D+2)y = \sin(3x)$

Auxiliary eqn : $m^2-3m+2=0$

$\Rightarrow (m-2)(m-1)=0$

$$y_c = Ae^{x}+Be^{2x}.$$

$$y_p = \frac{1}{D^2-3D+2}\sin(3x)$$

$$= \frac{1}{-9-3D+2}\sin(3x) = -\frac{1}{3D+7}\sin(3x)$$

$$= -\frac{(3D-7)}{9D^2-49}\sin(3x)$$

$$= \frac{1}{130}(3D-7)\sin(3x)$$

$$= \frac{1}{130}\left(9\cos(3x)-7\sin(3x)\right)$$

H.W

1. $(D^4-1)y = \sin x.$

2. $(D^2+4)y = \sin^2 x.$ $\qquad \frac{1}{D^2+4}\cos(2x)$

36.

$$\frac{1}{f(D)}x^m = \frac{1}{D^p(1+\phi(D))^n}x^m$$

$$= \frac{1}{D^p}(1+\phi(D))^{-n}x^m$$

$\downarrow$ Use binomial theorem for expansion.

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

$$(1\pm x)^{-1} = 1\mp x+x^2\mp x^3\cdots$$

$$(1\pm x)^{-2} = 1\mp 2x+3x^2\mp 4x^3+\cdots$$

Exm. $(D^4-1)y = x^4$

$$y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$$

$$y_p = \frac{1}{D^4-1} x^4 = -(1-D^4)^{-1} x^4$$

$$= -(1+D^4+D^8+D^{12}+\cdots)x^4$$

$$= -(x^4+24).$$

Exm. $(D^3+3D^2+2D)y = x^2$

A.E: $m(m^2+3m+2) = 0$

$\Rightarrow m = 0, -2, -1$

$$y_c = A + Be^{-x} + ce^{-2x}.$$

$$y_p = \frac{1}{D^3+3D^2+2D} x^2$$

$$= \frac{1}{2D\left(1+\frac{D^2+3D}{2}\right)} x^2$$

$$= \frac{1}{2D}\cdot\left[1 - \frac{3D+D^2}{2} + \left(\frac{3D+D^2}{2}\right)^2 - \cdots\right]x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3D+D^2}{2} + \frac{9D^2}{4} + \cdots\right]x^2$$

$$= \frac{1}{2D}\left[x^2 - 3x - 1 + \frac{9}{2}\right]$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{2}\right] = \frac{1}{2}\int\left(x^2-3x+\frac{7}{2}\right)dx$$

$$= \frac{1}{6}x^3 - \frac{3}{4}x^2 + \frac{7}{4}x.$$

Ex. $(D^2+3D+2)y = e^{2x}\sin x$

$$y_c = Ae^{-2x} + Be^{-x}.$$

$$y_p = \frac{1}{(D^2+3D+2)} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^2+3(D+2)+2} \sin x \quad [RS)$$

$$= e^{2x} \frac{1}{D^2+7D+12} \sin x$$

$$= e^{2x} \frac{1}{7D+11} \sin x = e^{2x} \frac{7D-11}{49D^2-121} \sin x$$

$$\cancel{= e^{2x} \frac{1}{11}\left(1+\frac{7}{11}D\right)^{-1} \sin x}$$

$$= -\frac{e^{2x}}{170}(7D-11)\sin x$$

$$\cancel{= \frac{e^{2x}}{11}\left[1-\frac{7}{11}D\right.}$$

$$= \frac{e^{2x}}{170}(11\sin x - 7\cos x)$$

H.W. $(D^2-1)y = \cosh x \cos x + 2^x$.

Method of Undetermined Co-efficients :-

$$f(D)y = X$$

When X is of particular form with special functions, we make a guess of the trial solution y_p.

Table

	X	y_p
1.	x^n or $a_0+a_1x+\cdots+a_nx^n$	$A_0+A_1x+\cdots+A_nx^n$.
2.	e^{ax}	Ae^{ax}.
3.	$a_nx^ne^{ax}$	$e^{ax}(A_0+A_1x+\cdots+A_nx^n)$
4.	$p\sin(ax)/q\cos(ax)$	$A\sin(ax)+B\cos(ax)$
5.	$pe^{bx}\sin(ax)$	$e^{bx}(A\sin(ax)+B\cos(ax))$
6.	$x^n\sin(ax)$	$(A_0+A_1x+\cdots+A_nx^n)\sin(ax)+(B_0+B_1x+\cdots+B_nx^n)\cos(ax)$

Ex^M.

$$(D^2+4)y = x^2$$

$$y_c = A\sin(2x) + B\cos(2x)$$

Let, the trial solution be $y_p = A_0 + A_1 x + A_2 x^2$

We have $(D^2+4)y_p = x^2$

$$\Rightarrow \quad 2A_2 + 4A_0 + 4A_1 x + 4A_2 x^2 = x^2$$

This is an identity. Comparing the co-eff, we set

$$A_2 + 2A_0 = 0$$

$$A_1 = 0$$

$$4A_2 = 1 \quad \Rightarrow A_2 = \frac{1}{4}$$

$$\Rightarrow A_0 = -\frac{1}{8}.$$

$$\therefore y_p = -\frac{1}{8} + \frac{1}{4}x^2. = \frac{1}{8}(2x^2-1).$$

Ex^M.

$$(D^2+1)y = 12\cos^2 x.$$

$$= 6(1+\cos(2x))$$

$$y_c = A\sin x + B\cos x$$

Let, the trial soln is $y_p = C_0 + C_1\cos(2x) + C_2\sin(2x)$

We have $(D^2+1)y_p = 6(1+\cos(2x))$

$$-4(C_1\cos(2x) + C_2\sin(2x)) + C_0 + C_1\cos(2x) + C_2\sin(2x)$$

$$= 6 + 6\cos(2x)$$

$$\Rightarrow \quad C_0 = 6, \quad -3C_1 = 6, \quad -3C_2 = 0$$

$$\Rightarrow C_1 = -2$$

$$\therefore y_p = 6 - 2\cos(2x)$$

A term of X is also a term of y_c.

$$(D-2)^2 y = e^{2x}$$

$$y_c = Ae^{2x} + Bxe^{2x}$$ $\underline{2}$: power of x.

Trial Solⁿ: $y_p = (A_0 + A_1 x + A_2 x^2)e^{2x}$.

A term of x is $x^r u$ and u is a term of y_c.

$$(D-2)^2 y = x^1 e^{2x}$$

$\underline{2+1}$: power of x

$$y_c = Ae^{2x} + Bxe^{2x}$$

Trial: $y_p = (A_0 + A_1 x + A_2 x^2 + A_3 x^3)e^{2x}$.

H.N.

1. $x^2 y'' - xy' - 3y = x^2 \ln(x)$.

2. $(D^4 - 9D^2)y = (x^2+1)\sin(3x)$