SCHOOL OF BASIC SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY BHUBANESWAR
Mathematics-I (MA1L001)

Tutorial Sheet-5

1. Vector algebra and gradients

Unless otherwise specified, $\mathbf{r}$ refers to the position vector $\mathbf{r} = (x, y, z)$, and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

(1) Consider the vectors $\mathbf{p} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j}$. Which of them are parallel to each other, and which of them are perpendicular to each other?

(2) If $\mathbf{a} = (2, -1, 2)$, $\mathbf{b} = (-1, 2, 1)$ and $\mathbf{c} = (1, -2, 1)$, find the following quantities:
- (a) $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}|$.
- (b) $\mathbf{a}.\mathbf{b}$, $\mathbf{a}.\mathbf{c}$ and $\mathbf{b}.\mathbf{c}$.
- (c) $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$.
- (d) The unit vector $\widehat{\mathbf{a}}$.
- (e) The angle between $\mathbf{b}$ and $\mathbf{c}$.
- (f) The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{c}$.
- (g) The component of $\mathbf{a}$ parallel to $\mathbf{b}$.
- (h) The component of $\mathbf{a}$ perpendicular to $\mathbf{b}$.

Be sure to type-check your answers: do not give a vector where a scalar is required, or *vice-versa*.

(3) Consider vectors $\mathbf{a} = (u, v, w)$ and $\mathbf{b} = (x, y, z)$. Give formulae for $\mathbf{a}.\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ and $|\mathbf{a}\times\mathbf{b}|^2$. Verify by direct expansion that

$$(\mathbf{a}.\mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2.$$

(4) Find grad(f) for the following functions
- (a) $f = x^2y + y^2z + z^2x$
- (b) $f = \sin(r)/r$
- (c) $f = e^{-x^2-y^2} + z$.

(5) If $f = x^2yz^3$ and $\mathbf{n} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, find the directional derivative $\mathbf{n}.\nabla(f)$.

(6) The scalar field f is given by $f = x\sin(xy) + z\cos(xy)$. Find the component of grad(f) in parallel to $(-1, 1, -1)$ at the point $(\pi/2, 2, 0)$.

(7) Put $f = x^2 - z^2$ and $g = 2xz + y^2$. Show that $\nabla(f)$ is always perpendicular to $\nabla(g)$.

(8) You are skiing on a mountain which happens to be the graph of the function $f(x, y) = 10 - x^2 - y^4$. You are at the point $(1, 1, 8)$. If you want to ski down the steepest path, what direction should you head?

(9) A fly is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly?

(10) Find a unit normal vector to the surface $xy^2 + 2yz = 8$ at the point $(3, -2, 1)$.

(11) Find the normal vector and the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point $(3, 4, 5)$.

(12) Find the angle between the surfaces $x \ln z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$.

2. Curvature

(1) Show that the curvature of a circle of radius a is $1/a$.

(2) Find the curvature of the twisted cubic $r(t) = (t, t^2, t^3)$ at a general point and at $(0, 0, 0)$.

(3) Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

3. Div and curl

(1) Find $\nabla.\mathbf{u}$ and $\nabla \times \mathbf{u}$ for the following vector fields:

$$\text{(a):} \quad \mathbf{u} = (xy, yz, 0) \qquad \qquad \text{(b):} \quad \mathbf{u} = (z, x, y).$$

(2) Consider a vector field of the form $\mathbf{u} = f(r)\mathbf{r}$, where f is a function of r only. Show that $\nabla.\mathbf{u} = 3\,f(r) + r\,f'(r)$. Show that if $\nabla.\mathbf{u} = 0$, then $f(r) = c/r^3$ for some constant c.
[**Hint:** remember the chain rule $\frac{\partial}{\partial x} f(r) = f'(r)\frac{\partial r}{\partial x}$.]

(3) Find constants a, b and c such that the vector field

$$\mathbf{v} = (x + 2y + az,\ bx - 3y - z,\ 4x + cy + 2z)$$

satisfies $\mathrm{curl}(\mathbf{v}) = 0$. For these values of a, b and c, find a potential function f with $\mathrm{grad}(f) = \mathbf{v}$.

(4) If $r = \sqrt{x^2 + y^2 + z^2}$, show that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

(5) Let Ω be a scalar field, and let $\mathbf{F}$ be a vector field. Show that
(a) $\mathrm{curl}(\Omega\mathbf{F}) = \Omega\ \mathrm{curl}(\mathbf{F}) - \mathbf{F} \times \mathrm{grad}(\Omega)$
(b) $\mathrm{curl}(\mathrm{grad}(\Omega)) = 0$.
Rewrite these identities in ∇ notation.

(6) Let $\mathbf{H}$ be a vector field that can be expressed as $\mathbf{H} = f\ \mathrm{grad}(g)$ for some scalar fields f and g. Show that $\mathbf{H}$ is perpendicular to $\mathrm{curl}(\mathbf{H})$ at every point.
[**Hint:** use the previous question.]
Now consider the vector field $\mathbf{H} = x^2y\,\mathbf{r}$ (where $\mathbf{r} = (x, y, z)$ as usual). Find scalar fields f and g such that $\mathbf{H} = f\ \mathrm{grad}(g)$. Calculate $\mathrm{curl}(\mathbf{H})$ and check directly that it is perpendicular to $\mathbf{H}$.

(7) For the vector field $\mathbf{A} = (x^2y,\ y^2z,\ z^2x)$, calculate
(a) $\nabla.\mathbf{A}$
(b) $\nabla(\nabla.\mathbf{A})$
(c) $\nabla \times \mathbf{A}$
(d) $\nabla \times (\nabla \times \mathbf{A})$
(e) $\nabla^2(\mathbf{A})$.
Verify the identity $\nabla^2(\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$ in this case.

(8) Show that for any vector fields $\mathbf{u} = (p,\ q,\ r)$ and $\mathbf{v} = (f,\ g,\ h)$ we have

$$\nabla.(\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}).\mathbf{v} - (\nabla \times \mathbf{v}).\mathbf{u}.$$