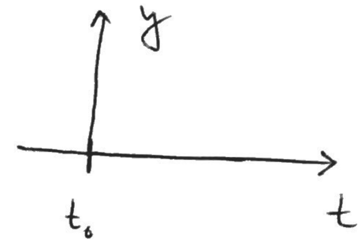
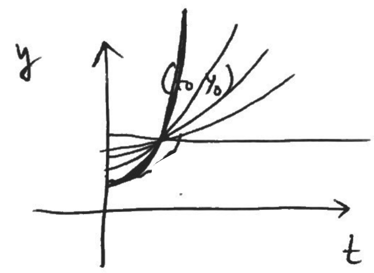
Picard Method :

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$



Goal : to find a particular soln which assumes the value y_0 at $t = t_0$.

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} f(t, y)\, dt$$



$$\Rightarrow \quad y = y_0 + \int_{t_0}^{t} f(t, y)\, dt$$

This is an integral eqn.

As a first approximation, y_1 to the soln, we put $y = y_0$ in $f(t, y)$ and integrate :

$$y_1 = y_0 + \int_{t_0}^{t} f(t, y_0)\, dt$$

then,

$$y_2 = y_0 + \int_{t_0}^{t} f(t, y_1)\, dt$$

and so on.

In general,

$$y_{k+1} = y_0 + \int_{t_0}^{t} f(t, y_k)\, dt, \quad k = 0, 1, 2, \ldots \quad \ldots (1)$$

ExM :

$$\frac{dy}{dt} = t + y, \quad y(0) = 1.$$

$$\therefore \quad y_1 = 1 + \int_0^t (1 + t)\, dt$$

$$= 1 + t + \frac{t^2}{2}.$$

$$\therefore \quad y_2 = 1 + \int_0^t \left(1 + 2t + \frac{t^2}{2}\right) dt = 1 + t + t^2 + \frac{t^3}{6}$$

$$y_3 = 1+t+t^2+\frac{t^3}{3}+\frac{t^4}{24}.$$

Analytic soln :

$$\frac{dy}{dt} - y = t$$

$$\text{I.F.} = e^{-t}$$

$$\Rightarrow \quad y(t) = 2e^{t} - t - 1$$
$$= 1+t+t^2+\frac{t^3}{3}+\frac{t^4}{12}+\frac{t^5}{60}+\cdots$$

Convergence :-

$$y = y_0 + \int_{t_0}^{t} f(t,y)\,dt =: G(y)$$

$\Rightarrow$ y is a fixed point of G.

If G is a contraction, the iteration (1) converges. Or if f satisfies (*), then $y_n \to y$.

Banach fixed pt. Theorem :-

Let, $G: \mathbb{R} \to \mathbb{R}$ be a contraction mapping, i.e.

$$|G(x) - G(y)| \leq K|x-y| \quad \text{with } K \in [0,1).$$

then, the sequence $x_n = G(x_{n-1})$ converges to the fixed pt x^* of G.

— An ODE can have no soln, unique soln, infinitely many soln.

Exm :

1. $y'^2 + y^2 + 1 = 0, \quad y(0) = 1$
2. $y' = 2x, \quad y(0) = 1$
3. $xy' = y-1, \quad y(0) = 1 \quad : \quad y = 1+\alpha x, \ \alpha \in \mathbb{R}$

Taylor Series Method :-

$$y' = f(t, y), \quad y(t_0) = y_0 \quad \cdots \quad (11)$$

Let, $y(t)$ is differentiable $(n+1)$-times in $[t_0, b]$. Then by Taylor's theorem,

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \cdots$$

$$+ \cdots + \frac{y^{(n)}(t_0)}{n!}(t - t_0)^n$$

$$+ \underline{\frac{y^{(n+1)}(c)}{(n+1)!}(t - t_0)^{n+1}}$$

for $c \in (t_0, b)$.

(Remainder)

Now, $y'(t_0) = f(t_0, y_0)$ is known.

$y''(t_0) = f_t + f_y y'$ " "

$y'''(t_0) = f_{tt} + 2 f_{ty} y' + f_{yy}(y')^2 + f_y y''$ is known.

Exm :- $y' = t^2 + y^2, \quad y(0) = 1.$

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \cdots$$

Now, $t_0 = 0, \quad y(t_0) = 1.$

$y'(t_0) = f(t_0, y_0) = 1.$

$y'' = 2t + 2yy'$

$\therefore y''(t_0) = 2$

$y''' = 2 + 2(y')^2 + 2yy'' \Rightarrow y'''(t_0) = 8.$

So, $y(t) = 1 + t + t^2 + \frac{8}{3!} t^3 + \cdots$

Now find $y(0.1)$ and $y(0.2)$.

H.W.

1. Solve $\frac{dy}{dx} = 2y + 3e^{x}$, $y(0) = 0$ by Picard & Taylor's Series and compute $y(0.2)$.

$$[\, y_T(0.2) = 0.8110 \,]$$

2. $y' = y^2 + 1$, $y(0) = 0$. Find $y(0.2)$.

3. $\frac{dy}{dx} = \log(xy)$, $y(1) = 2$. Find $y(1.1)$

$$[\, y(1.1) = 2.036)$$

Disadvantage :-

i) It is practically impossible to calculate infinite terms of Taylor Series in computer.

ii) Ambiguity about no of terms for correct result

iii) constructing new terms is tiresome.

Euler Method :-

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0 \quad \cdots (*)$$

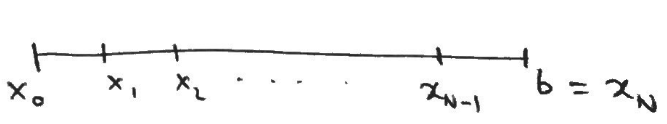
In this case, we will not find a differentiable fn $y(x)$, that satisfies $(*)$. Instead, a set of pts $\{(x_k, y_k)\}_k$ is generated as approximate values, i.e.

$$y(x_k) \simeq y_k.$$

Construction :-

Suppose we want solution at pts in $[x_0, b]$.

Then, $x_k = x_0 + hk$, $k = \frac{b - x_0}{N}$.



h : step size.

By Taylor's thm,

$$y(x_1) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(c)}{2!}(x-x_0)^2$$

$$c \in (x_0, x).$$

So that,

$$y(x_1) = y(x_0) + f(x_0, y_0)\,h + \frac{y''(c)}{2!}h^2$$

If h is very small, we can neglect $O(h^2)$.

Small oh or little oh : $o(h^2)$ $\quad f \in o(g)$ if $f/g \to 0$ as $x \to \infty$

Big oh : $O(h^2)$ (is of same order) $\quad f \in O(g)$ if $|f| \leq A g(x)\ \forall x \geq x_0$ for some const. A.

Exm $\quad x^2 \in o(x^3)$, $\quad x^2 \in O(x^2)$ as $x \to \infty$ $\quad \ln x \in o(x)$

and write

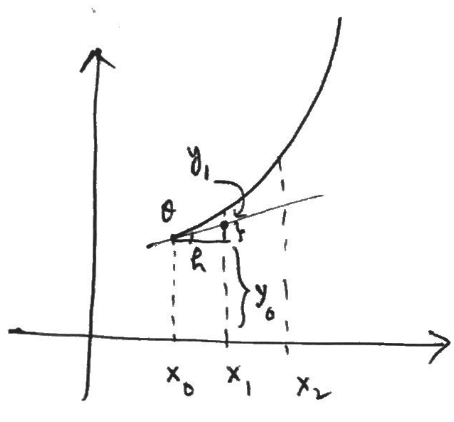
$$y(x_1) \simeq y_1 = y_0 + h f(x_0, y_0)$$

In general,

$$y_{k+1} = y_k + h f(x_k, y_k), \quad k = 0, 1, \ldots, N-1$$

Geometrical :

$$y_1 = y_0 + h \tan\theta$$
$$= y_0 + h \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$
$$= y_0 + h f(x_0, y_0)$$



A new eq$^{\underline{n}}$ $\quad \frac{dy}{dx} = f(x,y)$, $\quad y(x_1) = y_1$, being solved to get $y_2 = y_1 + h f(x_1, y_1)$. and so on.

Exm: $y' = Ry$ over $[0,5]$, $y(0) = y_0$.

By Euler approximation,

$$y_{k+1} = y_k + h f(x_k, y_k) \quad , \quad h = \frac{5}{N}$$

$$= y_k + hRy_k$$

$$= y_k(1+hR) \quad , \quad k = 0,1,\ldots,N-1$$

$\therefore$

$$y_1 = y_0(1+hR)$$

$$y_2 = y_1(1+hR) = y_0(1+hR)^2$$

$$\vdots$$

$$y_N = y_0(1+hR)^N$$

Take $R = 0.1$, $y_0 = 1000$.

If $N = 5$, $y(5) \simeq y_5 = 1000\left(1+\frac{0.1}{1}\right)^5 = 1610.51$

If $N = 60$, $y(5) \simeq y_{60} = 1000\left(1+\frac{0.1}{12}\right)^5 = 1645.31$

If $N = 1800$, $y(5) \simeq y_{1800} = 1000\left(1+\frac{0.1}{360}\right)^{1800}$

$$= 1648.61.$$

The exact value is: $y(5) = 1000\, e^{0.5} = 1648.72$.

So, it needs many iterations to get accurate result.

$\Rightarrow$ One needs modified method to get more accurate result in less effort.

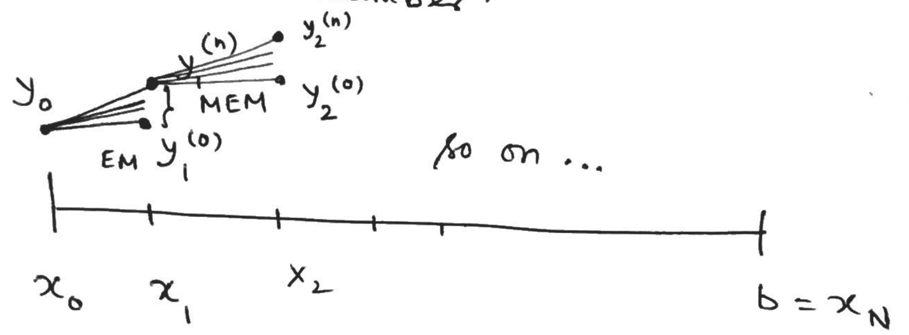
Modified Euler Method :-

Predictor step: $y_{k+1}^{(0)} = y_k + h f(x_k, y_k)$, $k = 0, 1, 2, \ldots$

Corrector step: $y_{k+1}^{(n)} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(n-1)})\right]$

$n = 1, 2, \ldots$

n : iteration number.



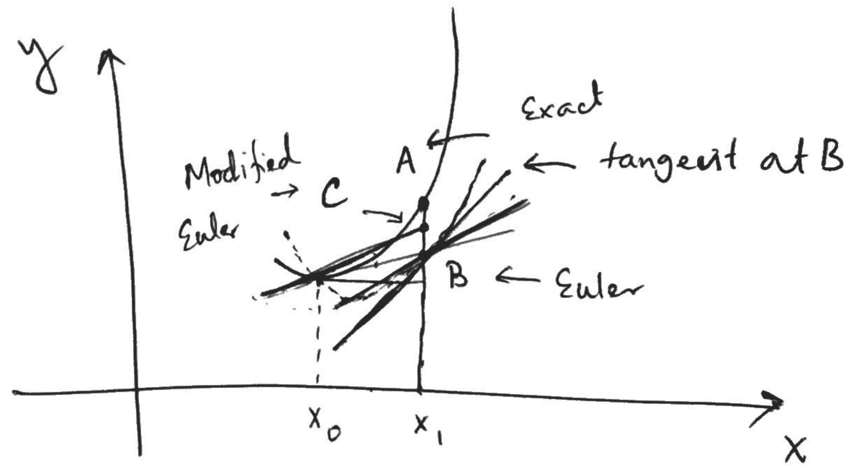
This method is also called Cauchy - Euler Method.

Stopping Criterion :-

Stop the iteration when

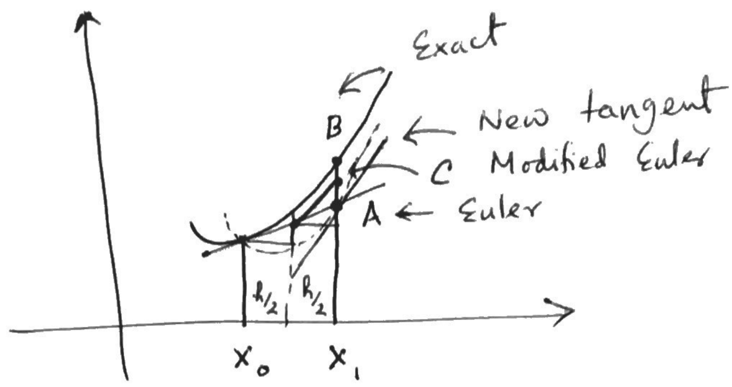
$$\left| y_k^{(n)} - y_k^{(n-1)} \right| < \text{tolerance Level.}$$

$(\sim 10^{-6})$.



Error per step: $O(h^3)$.

Graphical Representation :-



Runge-Kutta Method :-

Remedy to :

(I) Obtaining higher order derivative computation. (Taylor)

(II) for reasonable accuracy need very small step h. (Euler)

(III) integration of complicated functions (Picard)

IS :

- evaluation of $f(x,y)$ at few selective points in $[x_k, x_{k+1}]$.
- getting higher order accuracy without computing higher derivatives.

Taylor's Formula :

$$y(x_k+h) = y(x_k) + y'(x_k)h + y''(x_k)\frac{h^2}{2} + \frac{y'''(x_k)}{3!}h^3 + \frac{y^{(4)}(x_k)}{4!}h^4 + \frac{y^{(5)}(x_k)}{5!}h^5 + \cdots \quad (*)$$

RK2 :
We write

$$y_{K+1} = y_K + aK_1 + bK_2$$

where $K_1 = hf(x_K, y_K)$ &
$K_2 = h f(x_K + \alpha h,\ y_K + \beta K_1)$.

a, b, α, β are constants, determined by equating terms of (*).

So, $y_{K+1} = y_K + ahf(x_K, y_K) + bhf(x_K, y_K) + \alpha b h^2 f_x(x_K, y_K)$
$+ b\beta h^2 f(x_K, y_K) f_y + O(h^3)$ ... (**)

From (*) & (**) $\Rightarrow$

$$a + b = 1, \quad \alpha b = \frac{1}{2}, \quad \beta b = \frac{1}{2}.$$

Three eqns in four unknowns.

Case - I $\quad a = \frac{1}{2} = b, \quad \alpha = 1 = \beta.$

$$y_{K+1} = y_K + \frac{h}{2}\left[f(x_K, y_K) + f\left(x_K + h,\ y_K + hf(x_K, y_K)\right)\right]$$

This is Modified Euler Method. (2nd order RK)

Case - II $\quad a = \frac{2}{3}, \quad b = \frac{1}{3}, \quad \alpha = \frac{3}{2} = \beta.$

Case - III $\quad a = \frac{1}{4}, \quad b = \frac{3}{4}, \quad \alpha = \frac{2}{3} = \beta \quad$ (Ralston Method)
$\downarrow$
Minimizes truncation Error

Butcher Tableau :-

$K_1 = hf(x_K, y_K)$
$K_2 = hf(x_K + c_2 h;\ y_K + a_{21} K_1)$
$K_3 = hf(x_K + c_3 h,\ y_K + a_{31} K_1 + a_{32} K_2)$

$$y_{K+1} = y_K + b_1 K_1 + b_2 K_2 + b_3 K_3$$

is written in compact form as:

$$
\begin{array}{c|ccc}
0 & & & \\
c_2 & a_{21} & & \\
c_3 & a_{31} & a_{32} & \\
\hline
 & b_1 & b_2 & b_3
\end{array}
$$

← weights for k_i in y variable.

← weights

Weights for x-variable (nodes)

Modified Euler or Heun

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

RK4 :

The fourth order Runge-Kutta method (RK4) simulates the accuracy of Taylor's method of order 4.

We write

$$
\begin{aligned}
k_1 &= h f(x_k, y_k) \\
k_2 &= h f(x_k + c_2 h, y_k + a_{21} k_1) \\
k_3 &= h f(x_k + c_3 h, y_k + a_{31} k_1 + a_{32} k_2) \\
k_4 &= h f(x_k + c_4 h, y_k + a_{41} k_1 + a_{42} k_2 + a_{43} k_3)
\end{aligned}
$$

with $y_{k+1} = y_k + \sum_{i=1}^{4} b_i k_i$

A formal calculation and comparison to (*) gives a system of 11 equations in 13 unknowns.

The most popular choice is:

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 2/6 & 2/6 & 1/6
\end{array}
$$

i.e. $c_2 = \frac{1}{2}$, $a_{31} = 0$.

$k_1 = hf(x_k, y_k)$, $k_2 = hf(x_k + \frac{h}{2}, y_k + \frac{k_1}{2})$

$k_3 = hf(x_k + \frac{h}{2}, y_k + \frac{k_2}{2})$, $k_4 = hf(x_k + h, y_k + k_3)$

$$y_{k+1} = y_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(h^5)$$

This is RK4.

Exm :- $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$, $y(1) = 0$. Find $y(1.2)$, $y(1.4)$

$x_0 = 1$, $y_0 = 0$, $h = 0.2$.

$K_1 = hf(x_0, y_0) = 0.2 \frac{e}{1+e} = 0.1462$.

$K_2 = hf(x_0 + h/2, y_0 + k_1/2) = 0.2 f(1.1, 0.0731)$

$= 0.1402$

$k_3 = hf(x_0 + h/2, y_0 + \frac{k_2}{2}) = 0.2 f(1.1, 0.0701)$

$= 0.1399$

$K_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(1.2, 0.1399)$

$= 0.1348$

$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0.1462 + 0.2804 + 0.2798 + 0.1348)$

$= 0.1402$

So, $\underline{y(1.2) \simeq y_1 = 0.1402}$

Check: $y(1.4) \simeq y_2 = 0.2705$.

Higher Order equation / System of equations :-

The mth order differential equation:

$$y^{(m)} = f(x, y, y', \ldots, y^{(m-1)})$$

with $y(x_0) = A_1$, $y'(x_0) = A_2$, $\ldots$, $y^{(m-1)}(x_0) = A_m$.

Let, $y_1 = y$

$y_2 = y'$, $y_3 = y^{(2)}$, $\ldots$, $y_m = y^{(m-1)}$.

Then, $y_1' = y_2$, $y_2' = y_3$, $\ldots$, $y_{m-1}' = y_m$

and $y_m' = f(x, y, y_2, \ldots, y_m)$.

So, we can write as a system:

$$Y' = F(x, Y), \text{ with } Y(x_0) = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

RK2

$$Y_{K+1} = Y_K + \frac{1}{2}(K_1 + K_2)$$

with $K_1 = \begin{pmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{m1} \end{pmatrix}$ & $K_2 = \begin{pmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{m2} \end{pmatrix}$

where $K_{i1} = h f_i(x_K, A_1, A_2, \ldots, A_m)$

$K_{i2} = h f_i(x_K + h, A_1 + K_{11}, A_2 + K_{21}, \ldots, A_m + K_{m1})$

RK4

$$Y_{K+1} = Y_K + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where $K_J = \begin{pmatrix} K_{1J} \\ K_{2J} \\ \vdots \\ K_{mJ} \end{pmatrix}$ and K_{iJ}'s are as single ODE.

EXM:

$$y^{(3)} + 2y^{(2)} + y' + y = t$$

$$y(0) = 0 = y'(0), \quad y''(0) = 1$$

Let, $u_1 = y$, $u_2 = y'$, $u_3 = y^{(2)}$

Then, $u_1' = u_2$, $u_2' = u_3$, $u_3' = y^{(3)} = t - 2u_3 - u_2 - u_1$

$$\therefore \quad U' = \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \\ -u_1 - u_2 - 2u_3 + t \end{pmatrix} = F(t, u_1, u_2, u_3)$$

is the ODE. with initial condition

$$U(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

EXM:- $x'(t) = \overset{f_1}{\overset{\parallel}{x+2y}}, \quad y'(t) = \overset{f_2}{\overset{\parallel}{3x+2y}}, \quad x(0) = 6, \; y(0) = 4.$

Find $x(0.2)$, $y(0.2)$. $h = 0.2$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right), \quad x_0 = 6, \; y_0 = 4, \; t_0 = 0$$

with $K_1 = h \begin{pmatrix} 14 \\ 26 \end{pmatrix} = \begin{pmatrix} 2.8 \\ 5.2 \end{pmatrix}$

$$K_2 = h \begin{pmatrix} f_1\left(t_0 + \frac{h}{2}, x_0 + \frac{K_{11}}{2}, y_0 + \frac{K_{21}}{2}\right) \\ f_2\left(t_0 + \frac{h}{2}, x_0 + \frac{K_{11}}{2}, y_0 + \frac{K_{21}}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 4.12 \\ 7.08 \end{pmatrix}$$

$$K_3 = h \begin{pmatrix} f_1\left(t_0 + \frac{h}{2}, x_0 + \frac{K_{12}}{2}, y_0 + \frac{K_{22}}{2}\right) \\ f_2\left(t_0 + \frac{h}{2}, x_0 + \frac{K_{12}}{2}, y_0 + \frac{K_{22}}{2}\right) \end{pmatrix} = \begin{pmatrix} 4.63 \\ 7.85 \end{pmatrix}.$$

$$K_4 = h \begin{pmatrix} f_1\left(t_0 + h, x_0 + K_{13}, y_0 + K_{23}\right) \\ f_2\left(t_0 + h, x_0 + K_{13}, y_0 + K_{23}\right) \end{pmatrix} = \begin{pmatrix} 6.87 \\ 11.12 \end{pmatrix}$$

So, $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4.53 \\ 7.70 \end{pmatrix}$

$$= \begin{pmatrix} 10.53 \\ 11.70 \end{pmatrix}$$ (Ans)