Stability Analysis :-

Order of Accuracy:

Order of accuracy quantifies the rate of convergence of a numerical approximation of a DE to the exact solution.

The numerical approximate solution y_h is said to be nth order accurate if the error $E(h) = \|y - y_h\|$ follows:

$$E(h) \leq C h^n$$

$$\text{or} \quad E(h) = \mathcal{O}(h^n)$$

with respect to a norm $\|\cdot\|$.

Discretization Error:

Let, $\{(x_k, y_k)\}_{k=0}^{N}$ is the set of discrete approximations and $y = y(x)$ is the unique solution to the given problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

The global discretization Error e_k is defined by:

$$e_k = y(x_k) - y_k, \quad k = 0, 1, \ldots, N.$$

Note: $e_0 = 0$.

The local discretization error or truncation error ϵ_{K+1} is defined by

$$\epsilon_{K+1} = y(x_{K+1}) - \{y(x_K) + h\,\Phi(x_K, y(x_K))\}$$

$$K = 0, 1, 2, \ldots, N-1$$

It is the error committed in the single step from x_K to x_{K+1}. Here the method is:

$$y_{K+1} = y_K + h\,\Phi(x_K, y_K), \quad \Phi \text{ being called}$$

an increment function.

$$= \Psi(x_K, y_K, h).$$

Exm :-

Euler's Method:

$$|e_K| = |y(x_K) - y_K| = O(h)$$

$$|\epsilon_{K+1}| = |y(x_{K+1}) - y(x_K) - h f(x_K, y(x_K))| = O(h^2)$$

The error at the end of the interval is called the final global error.

$$E(h) = |y(b) - y_N| = O(h).$$

Exm :-

RK2

$$|e_K| = |y(x_K) - y_K| = O(h^2)$$

$$|\epsilon_{K+1}| = |y(x_{K+1}) - y(x_K) - h\,\Phi(x_K, y(x_K))| = O(h^3)$$

with

$$\Phi(x_K, y_K) = \frac{f(x_K, y_K) + f(x_{K+1}, y_K + h f(x_K, y_K))}{2}$$

Exm : RK4

$$|e_K| = |y(x_K) - y_K| = O(h^4)$$

$$|\epsilon_{K+1}| = |y(x_{K+1}) - y(x_K) - h\,\Phi(x_K, y(x_K))| = O(h^5)$$

Sometimes, e_{k+1} is denoted as local error and E_k as the global error, i.e. $e_{k+1} (= \epsilon_{k+1})$ & $E_k (= e_k)$ in our previous notation.

$\Rightarrow$ Three features of a numerical method are important while choosing a numerical method.

A. Convergent ? i.e. Is $E_k(h) \to 0$ as $h \to 0$
(global error)
does the approximated solution tends to the true soln as $h \to 0$.

B. Stable ? If we change initial condn y_0 to $\bar{y}_0$, are the computed solutions close, i.e.

$$|y_k - \bar{y}_k| \leq c\,|y_0 - \bar{y}_0|$$

$$y' = y, \; y(0) = 0 \;\Rightarrow\; y(x) = 0$$
$$y' = y, \; y(0) = \epsilon \;\Rightarrow\; y(x) = \epsilon e^x$$
$$\epsilon \neq 0 \;\Rightarrow\; |y(x)| \to \infty \text{ as } x \to \infty$$

NOT Stable.

C. Consistency ? Consistent if $e_k(h) \to 0$ as $h \to 0$
(local error)

Definition :- Ψ is consistent if for any $k \geq 0$

$$\lim_{h \to 0} \frac{e_k(h)}{h} = 0$$

Defn :- Ψ is of order p if $e_k(h) = O(h^{p+1})$

The basic convergence theorem for one-step solvers is that, if the local error is $O(h^{p+1})$, then the global error is $O(h^p)$. It is the behavior of the global error that dictates the notion of order of the numerical scheme.

B. <u>Stability</u> :-

Too complex $|y_k - \bar{y}_k| \leq c|y_0 - \bar{y}_0|$ in practical.

We thus talk about *linear stability*:

$y' = f(t, y)$ can be linearize as:

$$\frac{d}{dt}\left(y(t) - y_0\right) = f(t, y(t)) = f(t, y_0) + \frac{\partial f}{\partial y}(t, y_0)\,(y(t) - y_0) + o\left(|y(t) - y_0|\right)$$

So, the growth will be controlled by the term

$$\frac{\partial f}{\partial y}\,(y(t) - y_0).$$

So, it is sufficient to check stability for the linear equation $y' = \lambda y$, $\lambda \in \mathbb{C}/\mathbb{R}$.

<u>Definition</u> :- Suppose $y' = \lambda y$, $\lambda \in \mathbb{C}$. Then ~~the~~ numerical method Ψ is (linearly) stable if $y_k \to 0$ as $k \to \infty$.

<u>NOTE</u> : Stability of the original eqn is guranteed if $Re(\lambda) < 0$ $\left[\because \text{ the soln is } y(0)\, e^{\lambda t}\right]$

Exm :-

Stability of RK4.

Consider the linear eqn $y' = \lambda y = f(x, y)$

$1/2$	$1/2$			
$1/2$	0	$1/2$		
1	0	0	1	
	$1/6$	$2/6$	$2/6$	$1/6$

RK4 gives:

$$K_1 = hf(x_k, y_k) = h\lambda y_k$$

$$K_2 = hf\left(x_k + \frac{h}{2}, y_k + \frac{K_1}{2}\right) = h\lambda y_k\left(1 + \frac{h\lambda}{2}\right)$$

$$K_3 = hf\left(x_k + \frac{h}{2}, y_k + \frac{K_2}{2}\right) = h\lambda y_k\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{4}\right)$$

$$K_4 = hf(x_k + h, y_k + K_3) = h\lambda y_k\left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{4}\right)$$

$$\therefore \quad y_{k+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right) y_k$$

$$= \alpha y_k$$

The method ~~if~~ is absolutely stable if $|\alpha| \leq 1$

Stability region : $h\lambda \in (-2.78, 0)$

Relation between local & global error :-

General scheme :

$$y_{k+1} = y_k + h\Phi(x_k, y_k) \quad \cdots (*)$$

& $$e_{k+1} = y(x_{k+1}) - y(x_k) - h\Phi(x_k, y(x_k)) \quad \cdots (**)$$

Adding (*) & (**), we have,

$$e_{k+1} = y(x_{k+1}) - y_{k+1} - \left\{y(x_k) - y_k\right\} + h\left\{\Phi(x_k, y_k) - \Phi(x_k, y(x_k))\right\}$$

$$\Rightarrow \quad e_{k+1} = E_{k+1} - E_k + h\left\{\Phi(x_k, y_k) - \Phi(x_k, y(x_k))\right\}$$

$$\therefore \quad E_{k+1} = E_k + h\left\{\Phi(x_k, y(x_k)) - \Phi(x_k, y_k)\right\} + e_{k+1}$$

If Φ is Lipschitz, i.e.

$$|\Phi(x, y) - \Phi(x, z)| \leq L_\Phi |y - z|, \quad \forall y, z$$

then

$$|E_{k+1}| \leq |E_k| + hL|E_k| + |e_{k+1}|$$

$$\therefore \underline{|E_{k+1}|} \leq \underline{(1 + hL)|E_k| + |e_{k+1}|}.$$

This provides a bound on the global error in terms of local truncation error.