Boundary Value Problems :-

$$y'' = F(t, y, y') \qquad a \le t \le b$$

$$y(t_1) = y_1 \quad \text{and} \quad y(t_2) = y_2$$

This is called a two-point boundary value problem.

ExM :-

$$y'' = f(x), \quad y(0) = \alpha, \quad y(1) = \beta \qquad \text{(Dirichlet)}$$

$$y'' = f(x), \quad y'(0) = \alpha, \quad y'(1) = \beta \qquad \text{(Neumann)}$$

$$y'' = f(x), \quad y(0) = y(1) \qquad \text{(Periodic)}$$

ExM :- $y^{(3)} - y'' + y' - y = 0$, $y(0) = 1$, $y(1) = e$, $y'(2) = e^2$

The exact solution is : $y(x) = e^x$

ExM :- $y'' - 2y^3 = 0 \qquad y(1) = 1, \quad y'(2) + [y(2)]^2 = 0$

Soln : $y(x) = 1/x$.

Unlike IVP, BVP may or may not have unique solution.

Consider $y'' + y = 0$.

General solution is : $y(x) = A\cos x + B\sin x$

BCs : $y(0) = 1, \; y(\pi/2) = 0 \Rightarrow$ unique soln $y(x) = \cos x$

$y(0) = 0 = y(\pi) \Rightarrow$ infinite soln $y(x) = B\sin x$

$y(0) = y(\pi) = 1 \Rightarrow$ No solution.

Existence & Uniqueness :-

Consider the BVP:

$$(*) \ldots \begin{cases} y'' = f(x, y, y') , & x \in [a, b] \\ y(x_1) = y_1 , \ y(x_2) = y_2 \end{cases}$$

Let, $f(x, y, z)$ is continuous on the region $R = \{ (x, y, z) : x \in [a, b], y \in \mathbb{R}, z \in \mathbb{R} \}$ and f_x and f_y are continuous on R. If $\exists$ a constant $M > 0$ s.t.

$$f_y > 0 \quad \forall (x, y, z) \in R$$

$$\text{and } |f_z| \leq M \quad \forall (x, y, z) \in R$$

Then, (*) has a unique soln for $x \in [a, b]$.

$\Rightarrow$ This is a sufficient condition. Numerical methods for BVP are mainly divided into two classes:

A. Shooting method or Initial value method for BVP

B. Finite difference method (FDM).

A. Shooting method :- Since the number of condition at the 1st point are not sufficient to define the solution uniquely, we choose some condition at the 1st point arbitrarily to start the method for solving IVP. The solution is continued upto the 2nd point, where the condition will not be satisfied in general. Now the extra condition at the 1st pt is adjusted such that the condition at the 2nd pt is satisfied.

Suppose

$$y'' = p(x)\,y' + q(x)\,y + r(x) \quad , \quad y(x_1) = y_1 \; , \; y'(x_1) = 0$$

has unique soln. Moreover, let $Z(x)$ is the unique soln of

$$z'' = p(x)\,z' + q(x)\,z \quad \text{with} \quad \begin{aligned} z(y_1) &= 0 \\ z'(y_1) &= 1 . \end{aligned}$$

Then, the linear combination

$$u(x) = y(x) + c\,Z(x) \quad \text{is a soln of}$$

$$y'' = p(x)\,y' + q(x)\,y + r(x) .$$

Also,
$$u(x_1) = y(x_1) + c\,Z(x_1) = y_1$$
$$u(x_2) = y(x_2) + c\,Z(x_2) \; (= y_2) \Rightarrow$$

$$c = \frac{y_2 - y(x_2)}{Z(x_2)}$$

If $Z(x_2) \neq 0$, then the unique soln is:

$$u(x) = y(x) + \frac{y_2 - y(x_2)}{Z(x_2)}\, Z(x) \quad \text{is the}$$

unique soln.

Exm:-

$$y'' - (\lambda - 1)\,y' - \lambda y = -\lambda t - \lambda + 1 , \quad y(0) = 1, \quad y(1) = 1 + e^{-1}$$

$\lambda = 10$.

Exact soln $\quad y(t) = t + e^{-t}$

Using RK4, we find the soln of IVPs:

$$\begin{cases} y_1'' - (\lambda - 1)\,y_1' - \lambda y_1 = -\lambda t - \lambda + 1 \\ y_1(0) = 1, \quad y_1'(0) = 0 \end{cases}$$

and
$$\begin{cases} y_2'' - (\lambda - 1)\,y_2' - \lambda y_2 = 0 \\ y_2(0) = 0, \quad y_2'(0) = 1 \end{cases}$$

Consider $y(t) = y_1(t) + c \cdot y_2(t)$ where

$$\therefore \; c = \frac{(1+e^{-1}) - y_1(1)}{y_2(1)}.$$

Then, $y(0) = 1$, $y(1) = 1+e^{-1}$. If $h = 0.2$, then

$$y_1(1) \simeq y_1^5 \text{ by RK4}$$

and $y_2(1) \simeq y_2^5$ by RK4.

To determine $y(0.2)$ we use:

$$y(0.2) \simeq y_1^1 + \frac{(1+e^{-1}) - y_1^5}{y_2^5} \, y_2^1 .$$

B. Finite Difference Method :-

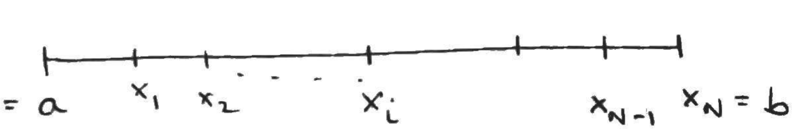
The shooting method is only for ODEs, FDM can also be applied to PDEs.

Consider the linear eqn:

$$(1) \quad \begin{cases} y'' = p(x)\, y' + q(x)\, y + r(x), & x \in [a,b] \\ \text{with BVs:} \quad a_0\, y(a) + b_0\, y'(a) = c_0 \\ \qquad\qquad\quad\; a_1\, y(b) + b_1\, y'(b) = c_1 \end{cases}$$

Step 1 :- Discretize the domain of the problem $[a,b]$.

Consider a partition $a = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = b$ into $(N+1)$ grid or nodal points to divide $[a,b]$ into N sub-intervals.



Denote $h = \dfrac{b-a}{N}$, called the step size. Then the node points are given by: $x_i = a + ih, \quad 0 \le i \le N$

A non-uniform partition is also possible. It is preferable if the solution of the BVP changes rapidly in some parts of $[a,b]$ as compared to other part.

Let, $p(x_i) = p_i$, $q(x_i) = q_i$, $r(x_i) = r_i$

and y_i is the numerical approximation of the true solution $y(x_i)$, $0 \le i \le N$.

Step 2 :- We now approximate the derivatives by divided differences.

By Taylor's series:

$$(11)\quad \begin{cases} y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y^{(3)}(x) + \cdots \\ y(x-h) = y(x) - h y'(x) + \frac{h^2}{2!} y''(x) - \frac{h^3}{3!} y^{(3)}(x) + \cdots \end{cases}$$

So, $$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h)$$

& $$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h)$$

give 1st order approximations. Also,

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

is called central difference approx. (2nd order)

For 2nd derivative,

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2)$$

Sometimes for the left boundary point, 2nd order method is required:

$$y(x+2h) = y(x) + 2h\,y'(x) + 2h^2 y''(x) + \frac{4}{3}h^3 y^{(3)}(x) + \cdots$$

We combine $y(x+h)$ and $y(x+2h)$ to eliminate h^2 term.

We have,

$$y'(x) = \frac{-3y(x) + 4y(x+h) - y(x+2h)}{2h} + O(h^2)$$

For a right Boundary pt, we can have

$$y'(x) = \frac{3y(x) - 4y(x-h) + y(x-2h)}{2h} + O(h^2).$$

These approximations can be derived from the formulas as well:

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \simeq \frac{y(x+h) - y(x)}{h}$$

$$y''(x) = \lim_{h\to 0} \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

We use these approximations to write the finite difference scheme for (1).

$$y'' = p(x)\,y' + q(x)\,y + r(x)$$

$$\Rightarrow \quad \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i + r_i \quad , \quad \underline{1 \le i \le N-1}$$

which can be rewritten as:

$$-\left(1 + \frac{h p_i}{2}\right) y_{i-1} + \left(2 + h^2 q_i\right) y_i + \left(\frac{1}{2} h p_i - 1\right) y_{i+1} = -h^2 r_i$$

$$1 \le i \le N-1$$

Step 3 :- This step is devoted to the treatment of the boundary conditions.

$$\text{(III)} \quad \begin{cases} a_0 y_0 + b_0 \dfrac{-3y_0 + 4y_1 - y_2}{2h} = c_0 \\[2ex] a_1 y_N + b_1 \dfrac{y_{N-2} - 4y_{N-1} + 3y_N}{2h} = c_1 . \end{cases}$$

i.e.

$$\left(a_0 - \frac{3b_0}{2h}\right) y_0 + \frac{4b_0}{2h} y_1 - \frac{b_0}{2h} y_2 = c_0$$

$$\frac{b_1}{2h} y_{N-2} - \frac{4b_1}{2h} y_{N-1} + \left(a_1 + \frac{3b_1}{2h}\right) y_N = c_1$$

We can write the system as:

$$MY = B \quad \cdots \quad (**)$$

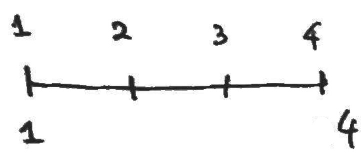
One then use numerical methods for linear system to solve for Y.

or $\underline{Y = M^{-1}B}$.

Here $Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{pmatrix}$

EXM : $\dfrac{d^2u}{dr^2} + \dfrac{1}{r}\dfrac{du}{dr} - \dfrac{u}{r^2} = 0 , \quad u(1) = 1, \quad u(4) = 1$

Use 4 nodes to approximate by FDM.



$$h = \frac{4-1}{3} = 1 .$$

$$r_i = 1 + (i-1)h, \quad i = 1(1)4 , \quad u(r_i) \simeq u_i$$

Finite difference scheme:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{1}{r_i}\frac{u_{i+1} - u_{i-1}}{2h} - \frac{u_i}{r_i^2} = 0 , \quad i = 2, 3 .$$

and,

$$u_1 = 1$$
$$u_4 = 1 .$$

$$\left(\frac{1}{h^2}+\frac{1}{2hr_i}\right)u_{i+1}-\left(\frac{2}{h^2}+\frac{1}{r_i^2}\right)u_i+\left(\frac{1}{h^2}-\frac{1}{2hr_i}\right)u_{i-1}=0.$$

$i=2,3$.

<u>$i=2$</u>

$$\left(1+\frac{1}{4}\right)u_3-\left(2+\frac{1}{4}\right)u_2+\left(1-\frac{1}{4}\right)u_1=0$$

$$\Rightarrow \frac{5}{4}u_3-\frac{9}{4}u_2+\frac{3}{4}u_1=0$$

<u>$i=3$</u>

$$\left(1+\frac{1}{6}\right)u_4-\left(2+\frac{1}{9}\right)u_3+\left(1-\frac{1}{6}\right)u_2=0$$

$$\Rightarrow \frac{7}{6}u_4-\frac{19}{9}u_3+\frac{5}{6}u_2=0$$

In matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{9}{4} & 5/4 & 0 \\ 0 & \frac{5}{6} & -\frac{19}{9} & 7/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\underline{AU=B}$$

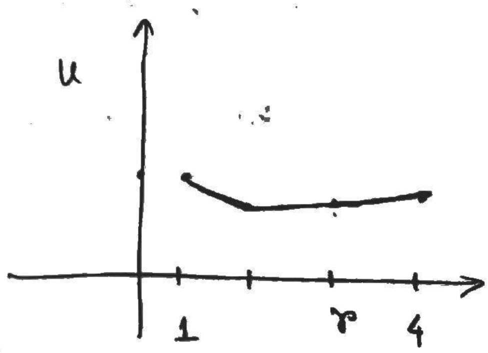
$$\therefore \underline{U=A^{-1}B}=\begin{pmatrix} 1 \\ 0.8202 \\ 0.8764 \\ 1 \end{pmatrix}$$

<u>Exact soln:</u>

$$u(r)=C_1r+\frac{C_2}{r} \quad \text{with}$$

$$u(1)=1=u(4)$$

i.e. $$u(r)=\frac{r}{5}+\frac{4}{5r}$$



$$\text{Exact soln} = \begin{pmatrix} 1 \\ 0.8 \\ 0.8667 \\ 1 \end{pmatrix}$$