Numerical Solutions of system of Linear equations :-

- Gauss-elimination
- LU decomposition
- Gauss - Jacobi
- Gauss - Seidel

A Linear system of n eqn in n unknowns is given by:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

In matrix notation,

$$AX = b, \quad \text{where}$$

$A = (a_{ij})_{n \times n}$ is called the co-efficient matrix.

Exm. Find the parabola $y = A + Bx + Cx^2$ that passes through $(1,1)$, $(2,-1)$ and $(3,1)$.

We obtain

$$A + B + C = 1$$
$$A + 2B + 4C = -1$$
$$A + 3B + 9C = 1$$

$$\therefore \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Let, $A = (\bar{a}_1\ \bar{a}_2 \ldots \bar{a}_n)$ where, $\bar{a}_i$'s are column ~~vec~~ vectors.

<u>Cramer's Rule</u> :-

For $\det(A) \neq 0$, the linear system $Ax = b$ has the unique solution:

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1(1)n$$

where, A_i is the matrix obtained from A, replacing a_i by b.

$\Rightarrow$ $A = 14/2 = 7$, $B = \frac{-16}{2} = -8$, $C = \frac{4}{2} = 2$.

$\Rightarrow$ Cramer's Rule is simple and elegant, but computationally a disaster.

(1) Let, R_n is the no of operations (Multiplication, addition) to calculate an $n \times n$ determinant.

Then,

$$R_n = n.R_{n-1} + n + (n-1)$$
$$= nR_{n-1} + (2n-1)$$

[determinant is expanded as a sum of $(n-1)$st order minors, multiplied by element]

Let, M_n be the no of operations in Cramer's Rule.

$$M_n = (n+1)R_n + n$$

[Computing $(n+1)$ det. & n divisions]

We have $R_2 = 3$

& $M_2 = 11$

Then, $R_n \sim O(n!)$ & $M_n \sim O((n+1)!)$ This is too large for large n.

For 8 eqns, we need to do 2540160 operations or 700 hrs. if one can perform one operation per sec.

(ii) Round-off error will be significant.

(iii) No information if $\det(A) = 0$.

Exm:

$$x + 2y + z = 1$$
$$x + 2y + z = 10$$

No soln

Exm:

$$\left.\begin{aligned} x + 2y + z &= 1 \\ 2x + 4y + 2z &= 2 \\ x + 2y &= 1 \end{aligned}\right\} \quad \det A = 0$$

but infinite soln

$x + 2y = 1$, $z = 0$ whole line

$(1-2c, c, 0)$ are soln $c \in \mathbb{R}$

Gauss Elimination :- (Direct Method)

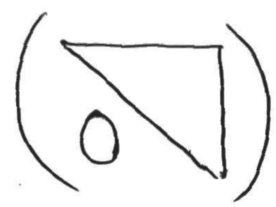
$$2x + 3y + 5z = 1$$
$$4y + z = 5$$
$$3z = 15$$

$$\Rightarrow \left.\begin{aligned} z &= 15/3 = 5 \\ y &= (5-5)/4 = 0 \\ x &= \frac{1-25}{2} = -12 \end{aligned}\right\} \text{Back-substitution}$$

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad : \text{upper triangular.}$$

Goal :- Gauss elimination converts any matrix A into an equivalent upper-triangular form.

$$Ax = b$$
$$\Updownarrow$$
$$Ux = b_1 \quad \text{(Solve by back substitution)}$$

$U =$ 

Exm :-

$n = 2$

$$\begin{cases} 2x + 3y = 5 & \cdots \text{(i)} \\ x + y = 3 & \cdots \text{(ii)} \end{cases} \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$A \quad X = b$$

$\text{(ii)} - \frac{1}{2} \times \text{(i)} \Rightarrow$

$$\begin{aligned} 2x + 3y &= 5 \\ -\tfrac{1}{2} y &= +1/2 \end{aligned} \Rightarrow \begin{pmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ +\frac{1}{2} \end{pmatrix}$$

Now use back-substitution to get

$$y = -1$$
$$x = \frac{5+3}{2} = 4$$

So, in matrix form:

$$[A \mid b] = \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 3 \end{array}\right) \xrightarrow{R_2 - \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

The matrix $[A \mid b]_{n \times (n+1)}$ is called the augmented matrix.

Note : $\frac{1}{2}$ (Multiplier) $= \dfrac{a_{21}}{a_{11}}$.

How are we getting $\begin{pmatrix} 2 & 3 \\ 0 & -1/2 \end{pmatrix}$ from $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$?

Ans: By pre-multiplying an elementary matrix.

Identity, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} = M_1$.

$$M_1 A = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{pmatrix} = U,$$ upper triangular

$$\therefore A = M_1^{-1} U = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & -1/2 \end{pmatrix}$$

$= LU$, where L is lower triangular & U is upper ".

This is called LU decomposition.

Exm:- $A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$R_2 + 2R_1$, $R_3 + R_1$

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & 1 \\ 0 & -2 & 5 & 4 \\ 0 & -2 & 3 & 2 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} -3 & 2 & -1 & 1 \\ 0 & -2 & 5 & 4 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

The solution is:

$z = 1$

$y = \dfrac{4-5}{-2} = \dfrac{1}{2}$

$x = -1/3$.

Check:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix}$$

$M_2 M_1 A = U$

$$\Rightarrow \quad A = M_1^{-1} M_2^{-1} U = LU.$$

with $\quad M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ & $M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\quad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

<u>LU decomposition Method</u> :-

Let, $A = LU$.

Need to solve : $\quad Ax = b$

$$\Rightarrow \quad LUx = b$$

Let, $\quad Ux = y$, then $Ly = b$

Forward substitution : $\quad Ly = b$

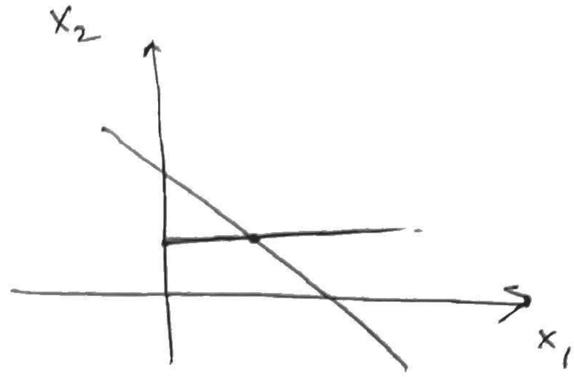
Backward substitution : $\quad Ux = y$

<u>Difficulty</u> :-

a) $\quad A = \begin{pmatrix} \boxed{0} & 1 \\ 1 & 0 \end{pmatrix}$: Gauss elimination fails.

(**) At the end :

e)
$$\epsilon x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$



Let, $\epsilon = 10^{-7}$

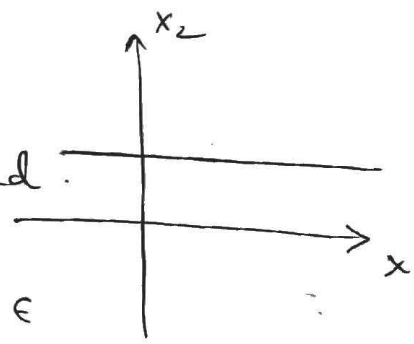
$$\left(\begin{array}{cc|c} \epsilon & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 - \frac{1}{\epsilon} R_1} \left(\begin{array}{cc|c} \epsilon & 1 & 1 \\ 0 & (1-\frac{1}{\epsilon}) & 2-\frac{1}{\epsilon} \end{array}\right)$$

The two eqns become

$$\left.\begin{aligned} \epsilon x_1 + x_2 &= 1 \\ (1-\tfrac{1}{\epsilon}) x_2 &= (2-\tfrac{1}{\epsilon}) \end{aligned}\right\}$$

i.e.
$$10^{-7} x_1 + x_2 = 1$$
$$(1-10^7) x_2 = (2-10^7)$$



The problem has become ill-conditioned.

This is because, the pivot element ϵ is very small, so it results very big round-off error.

Exm :-
$$\frac{1.2567}{0.0001} = 12567$$

Error Margin = 33.

$$\frac{1.26}{10^{-4}} = 12600$$

Remedy : To avoid these round-off error from small pivots, row interchanges are made (Partial Pivoting) or both rows and columns are interchanged (Complete Pivoting)

Exm:-
$$A = \begin{pmatrix} -2 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -8 & 4 \end{pmatrix}$$

We interchange rows to place the largest element of 1st column in the pivot.

$$A \xrightarrow{P_{12}} \begin{pmatrix} 6 & -6 & 7 \\ -2 & 2 & -1 \\ 3 & -8 & 4 \end{pmatrix} \xrightarrow[R_3 - \frac{1}{2}R_1]{R_2 + \frac{1}{3}R_1} \begin{pmatrix} 6 & -6 & 7 \\ 0 & 0 & 4/3 \\ 0 & -5 & 1/2 \end{pmatrix}$$

$$\xrightarrow{P_{23}} \begin{pmatrix} 6 & -6 & 7 \\ 0 & -5 & 1/2 \\ 0 & 0 & 4/3 \end{pmatrix} = U.$$

P_{12}, P_{23} are permulation matrices, so that

$$P_{23} M_1 P_{12} A = U$$

where $P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

Multiplication of M on the left by P interchanges rows, while multiplication of M on the right by P interchanges columns.

Now, $(P_{23} M_1 P_{23}) P_{23} P_{12} A = U$

$$\Rightarrow \quad PA = P_{23} M_1^{-1} P_{23} U$$

$$= P_{23} \begin{pmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} P_{23} U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ -1/3 & 1 & 0 \end{pmatrix} P_{23} U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} U = L.U.$$

Result :-

Let, $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Then there exists a permulation matrix P such that,

$$PA = LU$$

where L & U are lower and upper triangular matrices obtained from Gaussian elimination.

$\Rightarrow$ P is a permulation matrix $\Rightarrow$ $P^{-1} = P^T$.

$\boxed{**}$ b) Small Pivot element :-

$$0.0001\, x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Original soln. $x_1 = \dfrac{1}{1-0.0001}$, $x_2 = 2 - \dfrac{1}{1-0.0001}$

$$\left(\begin{array}{cc|c} 0.0001 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow[R_2 - 10^4 R_1]{} \left(\begin{array}{cc|c} 0.0001 & 1 & 1 \\ 0 & -9999 & -9998 \end{array}\right)$$

So, $x_2 = 0.9999 \simeq 1$

and $x_1 = 0$.

Operation Count :-

Amount of computation time $\equiv$ # of operations.

Gauss elimination :-

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow[R_3 - \frac{a_{31}}{a_{11}} R_1]{R_2 - \frac{a_{21}}{a_{11}} R_1} \begin{pmatrix} \square & \square \\ 0 & \boxed{} \\ 0 & \end{pmatrix}$$

~~Consider an elimination step with the pivot~~

In step i, we eliminate x_i from from $(n-i)$ equations, or $(n-i)$ rows. This needs $(n-i)$ divisions in computing the multiplier $m_{ji} \left(= \dfrac{a_{ji}}{a_{ii}}\right)$ and $(n-i)^2$ multiplications and as many substractions. Since we do $(n-1)$ steps, the total no of operations is

$$\sum_{i=1}^{n-1}\left[(n-i)+2(n-i)^2\right] \qquad n-i=k$$

$$= \sum_{k=1}^{n-1}\left[k+2k^2\right] = \frac{n(n-1)}{2} + \frac{n(n-1)(2n-1)}{3}$$

$$= \frac{2}{3}n^3 - \frac{n^2}{2} - \frac{n}{6} \equiv O(n^3)$$

So, Gauss elimination costs $O(n^3)$ operations.

Back-substitution :-

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & & \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$$

The solution x_i is given by

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij}x_j\right).$$

For x_i, we make $(n-i)$ multiplications and as many subtractions and 1 division. So total no of operations:

$$2\sum_{i=1}^{n}(n-i) + n = 2\sum_{k=1}^{n-1} k + n$$

$$= n(n-1)+n = n^2$$

Therefore, forward and backward substitution cost $O(n^2)$ operations.

When can we write $A = LU$? Is $\det(A) \neq 0$ enough?

EXM :-

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad \#$$

But, $\det(A) = 1 \neq 0$.

Result :- If determinants of all leading minors are non-zero, then $A = LU$. (or strictly column diagonally dominant)

Leading minors :

$A_1 = 1$

$A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $\boxed{\det(A_2) = 0}$

$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix} = A$, $\det(A_3) = 1$

Remedy : Interchange rows

$$A \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = U$$

$P_{23} M_1 A = U$ where

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$(P_{23} M_1 P_{23}) P_{23} A = U \Rightarrow \underline{P_{23} A} = (P_{23} M_1^{-1} P_{23}) U$$

$$= \underline{\underline{L . U.}}$$

with $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$$\boxed{PA = LU}$$

$\Rightarrow$ Let, $A = LU$ given. How can we find A^{-1}.

We know $A^{-1}A = AA^{-1} = I$.

Let, $A^{-1} = [b_1\ b_2 \dots, b_n]$, b_i's being the ith Column.

Then $A\,[b_1\ b_2 \dots b_n] = I = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ \vdots & 0 & \\ & \vdots & \end{pmatrix}_{n\times n}$

So, $Ab_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $Ab_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

So, solve for b_1 s: $LUb_1 = e_1$

Take $Ub_1 = Z_1$

$\Rightarrow LZ_1 = e_1$ forward subs.

then $Ub_1 = Z_1$ back subs.

and so on.

<u>Doolittle's Method</u> :-

$$A = LU = \begin{pmatrix} 1 & 0 & \dots & & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & 0 - & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & \dots & l_{nn-1} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & & \ddots & \\ 0 & & & u_{nn} \end{pmatrix}$$

Now solve for l_{ij} & u_{ij}'s. Diagonal elements of L are 1.

<u>Crout's Method</u>: If the diagonal elements of U are 1, it is called Crout's method.

<u>Why</u>:

$$A = LU = \begin{pmatrix} l_{11} & & 0 \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ 0 & & u_{33} \end{pmatrix}$$

12 unknowns 9 eqns : under-determined.

In general $n(n+1)$ " n^2 "

<u>NOTE</u>: Diagonally dominant $\Rightarrow$ leading minors non-zero $\Rightarrow A = LU$.

Exm. Use partial pivoting to write $PA = LU$ where

$$A = \begin{pmatrix} 4 & 3 & 6 \\ 6 & 2 & 1 \\ 5 & 5 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 & 6 \\ 6 & 2 & 1 \\ 5 & 5 & 5 \end{pmatrix} \xrightarrow{P_{12}} \begin{pmatrix} 6 & 2 & 1 \\ 4 & 3 & 6 \\ 5 & 5 & 5 \end{pmatrix}$$

$$\xrightarrow[R_3 - \frac{5}{6}R_1]{R_2 - \frac{4}{6}R_1} \begin{pmatrix} 6 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{16}{3} \\ 0 & \frac{10}{3} & \frac{25}{6} \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 6 & 2 & 1 \\ 0 & \frac{10}{3} & \frac{25}{6} \\ 0 & \frac{5}{3} & \frac{16}{3} \end{pmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2} \begin{pmatrix} 6 & 2 & 1 \\ 0 & \frac{10}{3} & \frac{25}{6} \\ 0 & 0 & \frac{39}{12} \end{pmatrix} = U$$

$$M_2 P_{23} M_1 P_{12} A = U$$

where

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}, \quad P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{5}{6} & 0 & 1 \end{pmatrix} \quad P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now

$$M_2 P_{23} M_1 P_{23} \, P_{23} P_{12} A = U$$

$$\Rightarrow \quad P_{23} P_{12} A = (P_{23} M_1 P_{23})^{-1} M_2^{-1} U$$

$$= (P_{23} M_1^{-1} P_{23} \, M_2^{-1}) U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 5/6 & 1 & 0 \\ 2/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} U$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 5/6 & 1 & 0 \\ 2/3 & \frac{1}{2} & 1 \end{pmatrix} U \quad \text{with } P^T P = I$$

Gauss - Jordan Method :-

$$A \xrightarrow[\text{elim.}]{\text{Gauss}} U \quad \text{so} \quad Ax = b \Rightarrow Ux = b' \text{ with back substitution.}$$

$$\big\downarrow \text{Gauss. Jordan}$$

$$(\text{often } I) = D \quad \text{so that} \quad Ax = b \Rightarrow Dx = b'' \text{ easier.}$$

$$A \longrightarrow \begin{pmatrix} 6 & 2 & 1 \\ 0 & \frac{10}{3} & \frac{25}{6} \\ 0 & 0 & \frac{39}{12} \end{pmatrix} \xrightarrow{R_1 - \frac{6}{10} R_2} \begin{pmatrix} 6 & 0 & -\frac{3}{2} \\ 0 & \frac{10}{3} & \frac{25}{6} \\ 0 & 0 & \frac{39}{12} \end{pmatrix}$$

$$\xrightarrow[R_2 - \frac{50}{39} R_3]{R_1 + \frac{6}{13} R_3} \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{10}{3} & 0 \\ 0 & 0 & \frac{39}{12} \end{pmatrix}$$

H.W. Solve $Ax = b$ using Gauss - Jordan Method.

$$A = \begin{pmatrix} 5 & 3 & 9 \\ 12 & 24 & 1 \\ 3 & 0 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}$$

H.W Solve the following system by Gauss - Jordan method

$$x + 2y - 3z = 2$$
$$6x + 3y - 9z = 6$$
$$7x + 14y - 21z = 13.$$

H.W Solve

$$2x + 4y + 6z = 18$$
$$4x + 5y + 6z = 24 \qquad \text{by Gauss - Jordan.}$$
$$3x + y - 2z = 4$$

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -8 \\ -3 & -5 & 8 \end{pmatrix}$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -8 & 0 & 1 & 0 \\ -3 & -5 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3+3R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1-2R_2]{R_3-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_2+2R_3]{R_1-R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 8 & -1 & 2 \\ 0 & 0 & 1 & 5 & -1 & +1 \end{array}\right)$$

So, $$A^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 8 & -1 & 2 \\ 5 & -1 & 1 \end{pmatrix}$$

H.W

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$