Iterative Methods :-

Gauss-elimination or LU decomposition techniques are direct methods, i.e. a solution is obtained after a single application of GE or LU. Once a solution is obtained, GE offers no method of refinement.

⇒ Iterative methods begin with an initial guess x_0, then obtain a series of improved approximations $x_1, x_2, \ldots$ that converge to the exact solution.

⇒ This method can be stopped as soon as the approx. x_n have converged to an acceptable precision. In direct method, bailing out early is not an option.

For large and sparse matrices, iterative methods are faster.

Jacobi Method :-

Exm:
$$7x_1 - x_2 = 6$$
$$x_1 - 5x_2 = -4$$

We rewrite the system as:

$$\left.\begin{aligned} x_1 &= \frac{1}{7}(6 + x_2) \\ \& \quad x_2 &= \frac{1}{5}(4 + x_1) \end{aligned}\right\} \cdots (1)$$

Then we iterate the approx. as:

$$x_1^{(k)} = \frac{1}{7}\left(6 + x_2^{(k-1)}\right)$$
$$x_2^{(k)} = \frac{1}{5}\left(4 + x_1^{(k-1)}\right)$$

for $k = 1, 2, \ldots$ with an initial approx $x_1^{(0)} = 0$, $x_2^{(0)} = 0$.

k	0	1	2	3	4
$x_1^{(k)}$	0	$6/7$	$34/35$	0.9959	0.9994
$x_2^{(k)}$	0	$4/5$	$34/35$	0.9943	0.9992

Clearly $x_1^{(k)} \to 1$, $x_2^{(k)} \to 1$, the exact soln.

Gauss-Seidel Method :-

One can modify Jacobi iteration to get faster (hopefully) convergence.

We iterate ① as follows:

$$x_1^{(k)} = \frac{1}{7}\left(6 + x_2^{(k-1)}\right) \quad \text{and then}$$

use the improved approx $x_1^{(k)}$ to calculate:

$$x_2^{(k)} = \frac{1}{5}\left(4 + x_1^{(k)}\right).$$ with an initial approx. $x_2^{(0)} = 0$. ($x_1^{(0)}$ does not matter)

Then, $x_1^{(1)} = 6/7$, $x_2^{(1)} = \frac{34}{35}$

$x_1^{(2)} = 0.9959$, $x_2^{(2)} = 0.9992$

k	0	1	2
$x_1^{(k)}$	0	$6/7$	0.9959
$x_2^{(k)}$	0	$34/35$	0.9992

Now try the system:

$$\left.\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6. \end{aligned}\right\}$$

So that

$$x_1 = 5x_2 - 4$$
$$x_2 = 7x_1 - 6$$

Thus Jacobi :

$$x_1^{(k)} = 5x_2^{(k-1)} - 4$$
$$x_2^{(k)} = 7x_1^{(k-1)} - 6.$$

or GS:

$$x_1^{(k)} = 5x_2^{(k-1)} - 4$$

$$x_2^{(k)} = 7x_1^{(k)} - 6$$

The table becomes: Jacobi

k	0	1	2	3	4
$x_1^{(k)}$	0	-4	-34	-174	-1244
$x_2^{(k)}$	0	-6	-34	-244	-1244

or GS

k	0	1	2
$x_1^{(k)}$	0	-4	-174
$x_2^{(k)}$	0	-34	-1224

i.e. both Jacobi & GS diverge.

Definition :- (Diagonally dominant)

An $n \times n$ matrix A is strictly diagnally dominant if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row.

i.e.

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for } i = 1, 2, \ldots, n.$$

Result :- If A is strictly diagonally dominant, then the system $Ax = b$ has a unique solution to which the Jacobi and the Gauss-Seidel method will converge for any initial approximation.

$$Ax = b \equiv \begin{pmatrix} 7 & -1 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

A is diag. dominant, but

$\begin{pmatrix} 1 & -5 \\ 7 & -1 \end{pmatrix}$ is not diag. dom.

In matrix notation:-

$$A = L + D + U. \quad \text{with } D = \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Jacobi Method:-

$$Dx = -(L+U)x + b$$

In iteration,

$$Dx^{(k)} = -(L+U)x^{(k-1)} + b$$

i.e. $$x^{(k)} = -D^{-1}(L+U)x^{(k-1)} + D^{-1}b$$

with initial guess $x^{(0)}$.

Gauss-Seidel Method:-

$$(D+L)x = -Ux + b$$

In iteration,

$$x^{(k)} = -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b$$

with initial guess $x^{(0)}$.

Algebraically,

Jacobi: $$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}\right). \quad i = 1, 2, \ldots, n$$

& GS: $$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)}\right).$$

Some observations :-

Jacobi :

$$x^{(k)} = -D^{-1}(L+U)\, x^{(k-1)} + D^{-1}b$$
$$= M_J\, x^{(k-1)} + D^{-1}b \quad , \text{ say.}$$

M_J is the Jacobi iteration matrix.

$$M_J = -D^{-1}(L+U) = D^{-1}(D-A)$$
$$= I - D^{-1}A$$

$$\therefore \quad I - M_J = D^{-1}A$$

$\Rightarrow$ A diag. dominant $\Rightarrow \|I - D^{-1}A\|_{L\infty} = \|M_J\|_{L\infty} = \max_i \left| \dfrac{\sum_{j\neq i} |a_{ij}|}{a_{ii}} \right| < 1.$

Proof

? $\Rightarrow$ $D^{-1}A$ is invertible $\Rightarrow$ A is invertible.

Let, $D^{-1}A$ is singular, then $\exists\, x \neq 0$ s.t

$$D^{-1}A x = 0$$

$$0 \neq \|x\| = \|(I - D^{-1}A)x\|_\infty \leq \|I - D^{-1}A\|_\infty \|x\|_\infty < \|x\|_\infty$$

which is absurd.

Gauss-Seidel :-

$$x^{(k)} = -(D+L)^{-1} U\, x^{(k-1)} + (D+L)^{-1} b$$
$$= M_G\, x^{(k-1)} + (D+L)^{-1} b.$$

M_G is the Gauss-Seidel iteration matrix.

$$M_G = -(D+L)^{-1} U$$
$$= -(D+L)^{-1}\left(A - (L+D)\right)$$
$$= I - (L+D)^{-1} A$$

Exm :-

$$4x_1 + x_2 + x_3 = 2$$
$$x_1 + 5x_2 + 2x_3 = -6$$
$$x_1 + 2x_2 + 3x_3 = -4$$

Exact Soln : $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

By Jacobi & Gauss-Seidel. (in matrix form).

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Jacobi's iteration :—

$$D x^{(k)} = -(A - D)\, x^{(k-1)} + b$$

$$\Rightarrow \quad x^{(k)} = (I - D^{-1}A)\, x^{(k-1)} + D^{-1} b$$

$$D^{-1}A = \begin{pmatrix} 1/4 & & 0 \\ & 1/5 & \\ 0 & & 1/3 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1/4 & \frac{1}{4} \\ \frac{1}{5} & 1 & 2/5 \\ \frac{1}{3} & 2/3 & 1 \end{pmatrix}$$

So, $$x^{(k)} = \begin{pmatrix} 0 & -1/4 & -1/4 \\ -\frac{1}{5} & 0 & -2/5 \\ -\frac{1}{3} & -2/3 & 0 \end{pmatrix} x^{(k-1)} + \begin{pmatrix} 2/4 \\ -6/5 \\ -4/3 \end{pmatrix}.$$

With zero initial guess,

$$x^{(1)} = \begin{pmatrix} 2/4 \\ -6/5 \\ -4/3 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 1.133 \\ -0.767 \\ -0.700 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 0.867 \\ -1.147 \\ -1.200 \end{pmatrix}$$

$$x^{(4)} = \begin{pmatrix} 1.087 \\ -0.893 \\ -0.858 \end{pmatrix}$$

Gauss-Seidel iteration:

$$(L+D)x^{(k)} = -Ux^{(k-1)} + b$$

$$\Rightarrow \quad x^{(k)} = -(L+D)^{-1}U\,x^{(k-1)} + (L+D)^{-1}b.$$

$$D+L = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 2 & 3 \end{pmatrix}, \quad (D+L)^{-1}U = \begin{pmatrix} 1/4 & 0 & 0 \\ -1/20 & 1/5 & 0 \\ -1/20 & -2/15 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/4 & 1/4 \\ 0 & -1/20 & 7/20 \\ 0 & -1/20 & -19/60 \end{pmatrix}$$

and
$$(D+L)^{-1}b = \begin{pmatrix} 1/4 & 0 & 0 \\ -1/20 & 1/5 & 0 \\ -1/20 & -2/15 & 1/3 \end{pmatrix} \begin{pmatrix} 2 \\ -6 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -13/10 \\ -19/30 \end{pmatrix}.$$

So,
$$x^{(k)} = \begin{pmatrix} 0 & -1/4 & -1/4 \\ 0 & 1/20 & -7/20 \\ 0 & 1/20 & 19/60 \end{pmatrix} x^{(k-1)} + \begin{pmatrix} 1/2 \\ -13/10 \\ -19/30 \end{pmatrix}.$$

With zero initial guess, the iteration will be:

$$x^{(1)} = \begin{pmatrix} 1/2 \\ -13/10 \\ -19/30 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 0.983 \\ -1.143 \\ -0.899 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 1.011 \\ -1.043 \\ -0.975 \end{pmatrix}$$

$$x^{(4)} = \begin{pmatrix} 1.004 \\ -1.011 \\ -0.994 \end{pmatrix}.$$

Both Jacobi & G.S iterative schemes can be written as:

$$x^{(k)} = P x^{(k-1)} + q \quad \text{- - -} (*)$$

where P is a constant matrix and q is a constant vector.

Jacobi: $P = M_J = I - D^{-1}A$

GS: $P = M_G = I - (L+D)^{-1}A$

<u>Convergence:</u>- The iterative scheme $(*)$ is convergent iff every eigen-value of P satisfies $|\lambda| < 1$, i.e. the spectral radius $\rho(P) < 1$.

$\Rightarrow$ If $\|P\| = \max\limits_{\|x\| \neq 0} \dfrac{\|Ax\|}{\|x\|} < 1$ for some norm, then $(*)$ will converge.

$$\& \quad x^* = P x^* + q$$

$$\Rightarrow \quad x^* - x^{(k)} = P(x^* - x^{(k-1)})$$

$$\Rightarrow \quad \underline{e_k = P e_{k-1}} \quad \Rightarrow \|e_k\| \leq \|P\| \, \|e_{k-1}\|$$

$\Rightarrow$ Jacobi:

$$P = -D^{-1}(L+U)$$

$$= - \begin{pmatrix} 1/a_{11} & & 0 \\ & \ddots & \\ 0 & & 1/a_{nn} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & \cdot & a_{1n} \\ \vdots & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & 0 \end{pmatrix}$$

A=diagonally dominant

$$= \begin{pmatrix} 0 & -a_{12}/a_{11} & \cdots & -a_{1n}/a_{11} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ -a_{n1}/a_{nn} & & & 0 \end{pmatrix} \quad \Rightarrow \sum_{j=1}^{n} |P_{ij}| < 1 \;\; \forall i$$

$$\Rightarrow \|P\|_\infty < 1.$$

Thm If $n=2$, Jacobi iteration converges iff GS converges.

Proof:

$$\rho(M_J) = \sqrt{\left|\frac{a_{21}a_{12}}{a_{11}a_{22}}\right|}$$

$$\& \ \rho(M_{GS}) = \frac{|a_{21}a_{12}|}{|a_{11}a_{22}|}$$

$$\rho(M_J) < 1 \quad \text{iff} \quad \rho(M_{GS}) < 1$$

Thm • For any $n > 2$, it is possible for Jacobi iteration to converge while GS diverges and conversely.

$$Ax = b \qquad A = B - C$$

$$Bx^{(k)} = Cx^{(k-1)} + b$$

$$\Rightarrow x^{(k)} = (B^{-1}C)\,x^{(k-1)} + B^{-1}b$$

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & & a \\ 1 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ b & & & & 1 & 1 \end{pmatrix} \quad \text{for } a, b \in \mathbb{R}.$$

For Jacobi, $\det(\lambda I - M_J) = 0$ gives

$$\lambda^n - ab\lambda^{n-2} + (-1)^{n+1}a = 0$$

For GS, we get

$$\lambda^{n-1}\left(\lambda - a\left(b + (-1)^n\right)\right) = 0$$

I) $a \geq 1,\ b = (-1)^{n+1} \Rightarrow \rho(M_{GS}) = 0,\ \rho(M_J) \geq 1$

II) $a = \frac{1}{2}(-1)^{n+1},\ b = (-1)^n \Rightarrow \rho(M_{GS}) = 1$ and $\rho(M_J) < 1$

Exm

$$A_J = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad A_J = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$\boxed{\rho(M_J) = 0.84 \quad \rho(M_{GS}) = 1}$

$\boxed{\rho(M_J) = 0.92 \quad \rho(M_{GS}) = 1}$

$A_J = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}$

$\boxed{\rho(M_J) = 0.71,\ \rho(M_{GS}) = 0.5}$

J conv.

GS div

$n > 2$

Exm

$$A_{GS} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad A_{GS} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$\boxed{\rho(M_J) = 1.27 \quad \rho(M_{GS}) = 0}$

$\boxed{\rho(M_J) = 1.32 \quad \rho(M_{GS}) = 0}$

$A_{GS} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

$\boxed{\rho(M_J) = 1;\ \rho(M_{GS}) = 1}$

J div

GS conv.

$n \geq 2$

Jacobi Algorithm :

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)} \right], \quad i = 1, 2, \ldots, n$$

Gauss - Seidel Algorithm :

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

$i = 1, 2, \ldots, n.$

Banded Matrix :-

If all matrix elements are zero outside a diagonally bordered band whose range is determined by K_1 & K_2:

$$a_{ij} = 0 \quad \text{if} \quad j < i - K_1 \quad \text{or} \quad j > i + K_2$$

$K_1, K_2 \geq 0$. then K_1 & K_2 are called the lower bandwidth and upper bandwidth respectively.

The bandwidth of the matrix is the max of K_1 & K_2.

$\Rightarrow$ A matrix is called banded if its bandwidth is reasonably small.

Exm Diagonal if $K_1 = K_2 = 0$

tri-diagonal if $K_1 = K_2 = 1$

upper-triangular if $K_1 = 0$; $K_2 = n-1$

Exm :-

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 4 & 1 & 3 & 0 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & -1 & 2 \end{pmatrix}_{4\times4}$$ is a banded matrix

with $K_1 = 1$, $K_2 = 1$. This matrix can be easily stored as:

$$A' = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 1 & 3 \\ 3 & -1 & 5 \\ -1 & 2 & 0 \end{pmatrix}_{4\times3}$$

Symmetric band Matrix :-

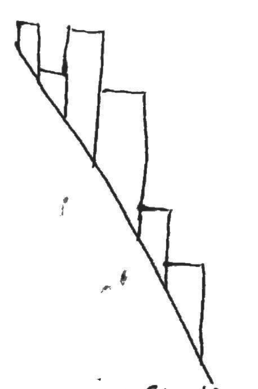
$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 1 & 4 & 0 \\ 0 & 4 & 2 & 5 \\ 0 & 0 & 5 & 1 \end{pmatrix}$$ is stored as:

$$A^* = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 2 & 5 \\ 1 & 0 \end{pmatrix}$$

Skyline method :-

If a sparse banded matrix is symmetric and has few zero entries, then it can be stored in single array.

$$A = \begin{pmatrix} 2 & 0 & 3 & 0 & 0 & 0 \\ & 1 & 0 & 5 & 0 & 0 \\ & & 5 & 0 & 0 & 0 \\ & \text{Sym} & & 6 & -1 & 0 \\ & & & & 3 & 2 \\ & & & & & 1 \end{pmatrix}$$ 6x6



Skyline

$$A' = [2 \ 1 \ 3 \ 0 \ 5 \ 5 \ 0 \ 6 \ -1 \ 3 \ 2 \ 1]$$

$$A'' = [1 \ 2 \ 5 \ 8 \ 10 \ 12]$$

Where the elements of A'' represent the element values of each column up to the diagonal position.

⇒ These methods significantly reduce the computer memory required to store the matrix entries.

⇒ In skyline solver, two one-dimensional matrices must be employed.