Numerical Methods for PDEs :-

BCM MathMethods

The general linear PDE of 2nd order in two variables is of the form:

$$a u_{xx} + 2b u_{xy} + c u_{yy} = F(x, y, u_x, u_y, u) . \quad \cdots (1)$$

$u(x,y)$ is an unknown function, F is a linear function of u_x, u_y and u.

Classifications :-

(1) is said to be

a) elliptic if $b^2 - ac < 0$

b) parabolic if $b^2 - ac = 0$

c) hyperbolic if $b^2 - ac > 0$.

Exm :- 1. $u_{xx} - 4u_{xy} + 4u_{yy} = e^y$

$a = 1, \quad b = -2, \quad c = 4 \quad \Rightarrow \quad b^2 - ac = 4 - 4 = 0.$

The eq^n. is parabolic

2. $u_{xx} + (1+y^2) u_{yy} - 2y(1+y^2) u_y = 0$

$a = 1, \quad b = 0, \quad c = (1+y^2) \quad \Rightarrow \quad b^2 - ac = -(1+y^2) < 0$

The eq^n. is elliptic

3. $u_{tt} - c^2 u_{xx} + u_x = 0$

$a = 1, \quad b = 0, \quad c = -c^2 \quad \Rightarrow \quad b^2 - ac = c^2 > 0$

The eq^n. is hyperbolic

4. $u_{yy} - y u_{xx} = 0$

$a = -y, \quad b = 0, \quad c = 1 \quad \Rightarrow \quad b^2 - ac = y$

on x-axis $(y = 0)$ parabolic, for $y > 0$, hyperbolic and for $y<0$, elliptic.

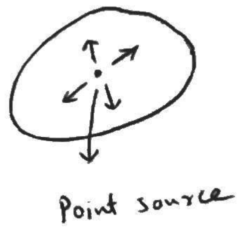
$\Rightarrow$ The behavior of the solution differs from type to type, so do the additional condn.

Model Problems :-

Elliptic :

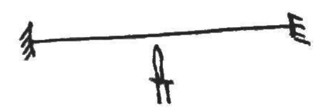
Laplace eqn. $\quad u_{xx} + u_{yy} = 0$

Poisson eqn. $\quad u_{xx} + u_{yy} = + f(x,y)$



Parabolic :

Heat eqn. $\quad u_t - u_{xx} = f(x,t)$



Hyperbolic :

Wave eqn. $\quad u_{tt} - u_{xx} = f(t,x)$

Analytical Methods to solve :

- Laplace / Fourier transform
- Seperation of variables.

Numerical Methods :

1) Laplace eqn. / Poisson : Finite diff. method
2) Heat eqn. : Bender Schmidt, Crank-Nicholson, Du-Fort Frankel methods
3) Wave eqn. : Explicit method / FDM.

Elliptic PDEs :-

Laplace / Poisson equation:

$$u_{xx} + u_{yy} = f(x,y) \quad , \quad x, y \in \underline{\Omega \subseteq \mathbb{R}^2}$$

In case of Laplace eq^n $f(x,y) = 0$.

Notation:

$$\left.\begin{aligned} \nabla &\equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) && (\text{nabla}) \\ \nabla \cdot \nabla \equiv \nabla^2 \equiv \Delta &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} && (\text{Delta}) \end{aligned}\right\} \text{operators}$$

1D Poisson Problem :-

$$\begin{cases} u_{xx}(x) = f(x) & , \quad x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

FDM

(step 1)

S1 Sub-divide $(0,1)$ into n equal sub-intervals

$\{x_0 = 0, x_1, \ldots, x_n = 1\}$ with

$$h = \frac{1}{n} \quad (\text{mesh size})$$

and $x_i = ih, \quad i = 0, \ldots n.$

Let, $u(x_i) \simeq u_i$, the approximate value.

Now $u_{xx}(x) = f(x)$.

$$\Rightarrow \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i) \quad , \quad \underline{1 \le i \le n-1}$$

and $u_0 = 0 = u_n$

$$\frac{1}{h^2} u_{i+1} - \frac{2}{h^2} u_i + \frac{1}{h^2} u_{i-1} = f(x_i)$$

S2 This can be rewritten as:

$$AU = b \quad \text{where}$$

$$A = \begin{pmatrix} -2/h^2 & 1/h^2 & 0 & \cdots & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & & & \\ 0 & & & 1/h^2 & -2/h^2 \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{n-1}) \end{pmatrix}$$

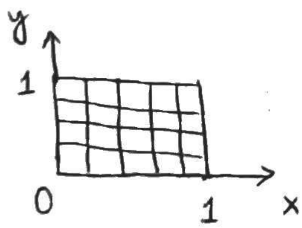
$$= \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & \\ 1 & -2 & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & & & 1 \\ 0 & & & 1 & -2 \end{pmatrix}_{(n-1)\times(n-1)}$$

H.W.

$$u_{xx} = -(3x + x^2)e^x, \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

Take $n = 5$. $h = 1/5$. $x_0 = 0$ ——— $1 = x_5$

2D Poisson equation:-



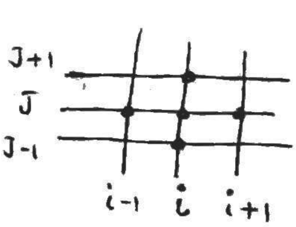
$$\nabla^2 u(x,y) = f(x,y), \quad (x,y) \in \Omega = [0,1]^2$$

$$\left.\begin{array}{l} u(0,y) = g(y), \quad u(1,y) = 0, \\ u(x,0) = 0, \quad u(x,1) = 0 \end{array}\right\} \text{Dirichlet BCs}$$

S1 $\Delta x = \dfrac{1-0}{n}$, sub-dividing $(0,1)$ into n

$\Delta y = \dfrac{1-0}{m}$ sub-dividing $(0,1)$ into m

sub-intervals.



$x_i = i\Delta x, \quad y_J = J\Delta y.$ $\quad i = 0(1)n$, $J = 0(1)m$.

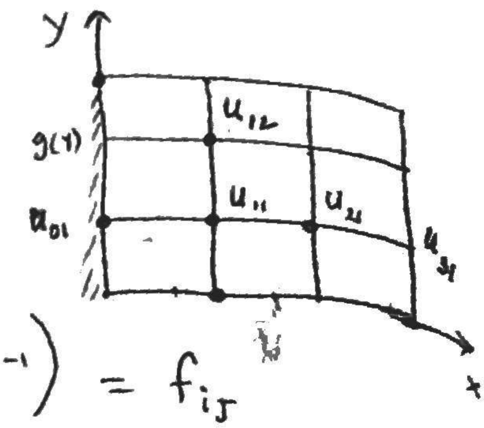
$$u_{xx} + u_{yy} = f(x,y)$$

$$\Rightarrow \frac{u_{i+1,J} - 2u_{iJ} + u_{i-1J}}{\Delta x^2} + \frac{u_{iJ+1} - 2u_{iJ} + u_{iJ-1}}{\Delta y^2} = f(x_i, y_J)$$

$1 \le i \le n-1, \ 1 \le J \le m-1$

For convenience, let $\Delta x = \Delta y = h$

$$n = m = 3$$



$$\frac{1}{h^2}\left(u_{i+1,j} + u_{i,j+1} - 4u_{ij} + u_{i-1,j} + u_{i,j-1} \right) = f_{ij}$$

$1 \le i \le 2$
$1 \le j \le 2$

S2

In matrix form,

$$AU = b$$

where

$$A = \frac{1}{h^2}\left(\begin{array}{cc|cc} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ \hline 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{array}\right)_{4\times4} \quad 4\times4 = (n-1)(m-1) \times (n-1)(m-1)$$

$$U = \begin{pmatrix} u_{11} \\ u_{21} \\ \hline u_{12} \\ u_{22} \end{pmatrix}_{4\times1} \qquad b = \begin{pmatrix} f_{11} - g(y_1) \\ f_{21} \\ \hline f_{12} - g(y_2) \\ f_{22} \end{pmatrix}$$

Try $n=4,\ m=4$

$\vdots$

S3 Then solve $AU=b$ by Gauss-elimination / LU / iterative methods.

H.W. Solve the poisson eqn. $u_{xx} + u_{yy} = -81xy$, $0<x<1$, $0<y<1$

given that $u(0,y)=0$, $u(x,0)=0$, $u(1,y)=100$

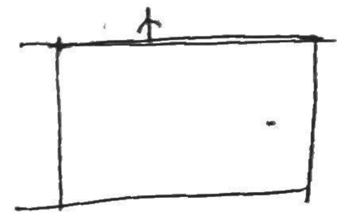
$u(x,1)=100$ and $h = 1/3$.

$AU = b$ + Gauss-Seidel $\equiv$ Liebmann's Method.

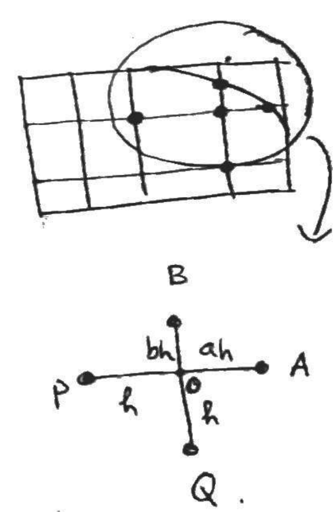
Neumann BCs

$$\begin{cases} u_{xx} + u_{yy} = f(x,y) \\ u(0,y) = 0, \quad u(1,y) = 0 \\ u(x,0) = 0 \quad \dfrac{\partial u}{\partial y}(x,1) = 0 \end{cases}$$

$$\frac{u_{i,m} - u_{i,m-1}}{h} = 0. \qquad 1 \le i \le n-1$$



Irregular boundary :-



By Taylor's,

$$u_A = u_0 + ah\frac{\partial u_0}{\partial x} + \frac{(ah)^2}{2!}\frac{\partial^2 u_0}{\partial x^2} + \cdots$$

$$u_P = u_0 - h\frac{\partial u_0}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u_0}{\partial x^2} + \cdots$$

We ignore $O(h^3)$ terms for small h.

$$\Rightarrow \quad u_A + a u_P \simeq (1+a)u_0 + \frac{1}{2}a(a+1)h^2\frac{\partial^2 u_0}{\partial x^2}$$

$$\therefore \quad \frac{\partial^2 u_0}{\partial x^2} = \frac{2}{h^2}\left[\frac{1}{a(1+a)}u_A + \frac{1}{(1+a)}u_P - \frac{1}{a}u_0\right]$$

$||^{ly}$, $$\frac{\partial^2 u_0}{\partial y^2} = \frac{2}{h^2}\left[\frac{1}{b(1+b)}u_B + \frac{1}{1+b}u_Q - \frac{1}{b}u_0\right]$$

$$\therefore \quad \nabla^2 u_0 = 0$$

$\Rightarrow$ we get five-points Stencil., let $a = \frac{1}{2} = b$,

Then the stencil looks like $\left\{\begin{matrix} & 4/3 & \\ 4/3 & -4 & 4/3 \\ & 4/3 & \end{matrix}\right\}$ instead $\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}$