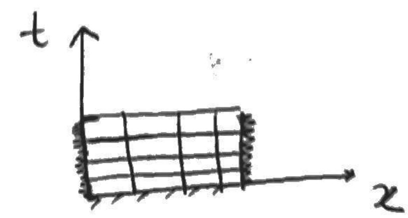
Parabolic PDEs :-

$$\begin{cases} u_t - \nu u_{xx} = f(x,t), & 0 \le x \le \ell,\ t > 0 \\ u(x,0) = u_0(x) & \text{IC} \\ u(0,t) = u(\ell,t) = 0 & \text{BC} \end{cases}$$

ν : diffusivity constant.



Bender - schmidt Method :-

Consider a mesh in xt-plane as follows:

$$\Delta x = \frac{\ell}{N}$$

$$\Delta t = \frac{T}{m}.$$

and $x_i = i\Delta x$, $\quad 0 \le i \le N$.

& $t_n = n\Delta t \quad 0 \le n \le m$

$$u(x_i, t_n) \simeq u_i^n, \quad f(x_i, t_n) = f_i^n$$

Then, $\quad u_t - \nu u_{xx} = f(x,t)$

$$\Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} - \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = f_i^n.$$

$\Rightarrow$ Let, $\lambda = \dfrac{\nu \Delta t}{\Delta x^2}$

$$\Rightarrow \quad u_i^{n+1} = (1-2\lambda)u_i^n + \lambda u_{i+1}^n + \lambda u_{i-1}^n + \Delta t f_i^n$$

$1 \le i \le N-1,$

This is an explicit formula. $\quad \underline{n \ge 0}$.

$\Rightarrow$ This method is valid when $\lambda \le \frac{1}{2}$ (convergence)

(Courant number)

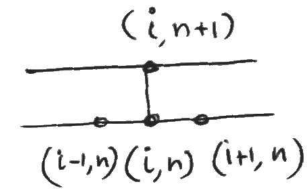
CFL condition

In matrix form,

$$\underline{U^{n+1} = AU^n + b}$$

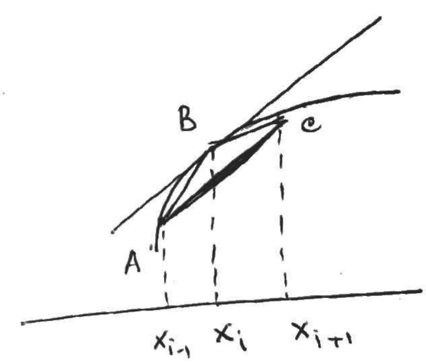
where $U^{n+1} = \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{pmatrix}$, $A = \begin{pmatrix} (1-2\lambda) & \lambda & 0 & \cdots & \\ \lambda & (1-2\lambda) & \lambda & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & & \lambda & (1-2\lambda) \end{pmatrix}$

and $b = \begin{pmatrix} \lambda u_0^n + \Delta t\, f_1^n \\ \Delta t\, f_2^n \\ \vdots \\ \Delta t\, f_{N-2}^n \\ \lambda u_N^n + \Delta t\, f_{N-1}^n \end{pmatrix}$



This method is called Bender-Schmidt method.

FDM



$u'(x_i) \simeq$ slope of AB

$= \dfrac{u_i - u_{i-1}}{\Delta x}$

or $\simeq$ slope of BC

$= \dfrac{u_{i+1} - u_i}{\Delta x}$

or $\simeq$ slope of AC $= \dfrac{u_{i+1} - u_{i-1}}{2\Delta x}$.

Crank-Nicolson Method :—

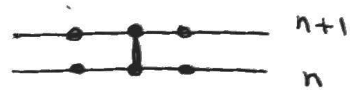
The CFL condition is a severe restriction on Δt. For example, $\nu = 1$, if $\Delta x = 0.1$, then $\Delta t \leq 0.005$. This condition can be avoided by using implicit scheme or Crank-Nicolson method (CN).

CN scheme :-

$$u_t - \nu u_{xx} = f(x,t).$$

$$\Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\nu\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right] + f_i^n$$

Let, $\lambda = \frac{\nu \Delta t}{\Delta x^2}$

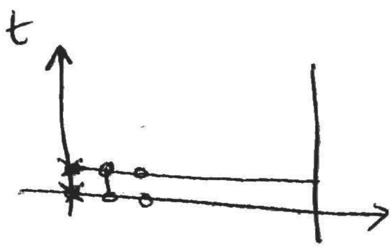


$$\Rightarrow \quad (1+\lambda)u_i^{n+1} - \frac{\lambda}{2}u_{i-1}^{n+1} - \frac{\lambda}{2}u_{i+1}^{n+1} = \frac{\lambda}{2}u_{i-1}^n + \frac{\lambda}{2}u_{i+1}^n + (1-\lambda)u_i^n + \Delta t f_i^n$$

$$1 \le i \le N-1, \quad n \geq 0$$

This is called Crank-Nicolson scheme. This is convergent for all λ. So, for a certain Δx, Δt can be taken arbitrarily, without restriction.

In matrix form,



$$AU^{n+1} = BU^n + b, \quad \underline{n \geq 0}.$$

where

$$U^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{pmatrix},$$

$$A = \begin{pmatrix} 1+\lambda & -\lambda/2 & 0 & \cdots & 0 \\ -\lambda/2 & 1+\lambda & -\lambda/2 & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & & -\frac{\lambda}{2} & 1+\lambda \end{pmatrix}, \quad B = \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & 1-\lambda & \frac{\lambda}{2} & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & & \frac{\lambda}{2} & 1-\lambda \end{pmatrix}$$

$$b = \begin{pmatrix} \frac{\lambda}{2}(u_0^n + u_0^{n+1}) + \Delta t f_1^n \\ \Delta t f_2^n \\ \vdots \\ \Delta t f_{N-2}^n \\ \frac{\lambda}{2}(u_N^n + u_N^{n+1}) + \Delta t f_{N-1}^n \end{pmatrix}$$

In another way, it can be interpreted as:

$$-\frac{\lambda}{2}u_{i-1}^{n+1} + (1+\lambda)u_i^{n+1} - \frac{\lambda}{2}u_{i+1}^{n+1} = \hat{u}_i^{n+1}$$

and $$\hat{u}_i^{n+1} = \frac{\lambda}{2}u_{i-1}^{n} + (1-\lambda)u_i^n + \frac{\lambda}{2}u_{i+1}^n + \Delta t\, f_i^n$$

so that $\hat{u}_i^{n+1}$ — predictor (explicit method)

u_i^{n+1} — corrector (implicit method).

DuFort-Frankel Method :-

$$u_t - \nu u_{xx} = f(x,t)$$

By central FDM,

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \nu\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = f_i^n$$

Now, $2u_i^n$ is replaced by $(u_i^{n-1} + u_i^{n+1})$ to get

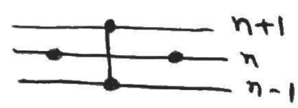
$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \nu\frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2} = f_i^n$$

Let, $\mu = \dfrac{2\nu\Delta t}{\Delta x^2}$ $\qquad 1 \le i \le N-1$, $n \ge 1$.

So, $$(1+\mu)u_i^{n+1} = (1-\mu)u_i^{n-1} + \mu(u_{i-1}^n + u_{i+1}^n) + f_i^n$$

i.e. $$u_i^{n+1} = \frac{1-\mu}{1+\mu}u_i^{n-1} + \frac{\mu}{1+\mu}(u_{i-1}^n + u_{i+1}^n) + \frac{1}{1+\mu}f_i^n$$

This is an explicit scheme, but unconditionally stable for all $\mu > 0$.



In matrix form, it can be written as:

$$U^{n+1} = A\bar{U}^{n}_{n-1} + b \quad , \quad \underline{n \geq 1}.$$

where,

$$A = \left(\begin{array}{cccc|cccc} r & 0 & \cdots & 0 & 0 & s & 0 \cdots & 0 \\ 0 & r & 0 \cdots & 0 & s & 0 & s \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \ddots & \\ 0 & \cdots & \cdots & r & 0 & \cdots & \cdots & s\;0 \end{array}\right), \quad r = \frac{1-\mu}{1+\mu}, \quad s = \frac{\mu}{1+\mu}.$$

$$\bar{U}^{n}_{n-1} = \begin{pmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ u_{N-1}^{n-1} \\ \hline u_1^{n} \\ \vdots \\ u_{N-1}^{n} \end{pmatrix}, \quad b = \begin{pmatrix} s u_0^n + \frac{1}{1+\mu} f_1^n \\ \frac{1}{1+\mu} f_2^n \\ \vdots \\ s u_N^n + \frac{1}{1+\mu} f_{N-1}^n \end{pmatrix}$$

For $n=1$, i.e. first time step, the solution is calculated by CN or Backward-Euler, central difference scheme. and then for $n \geq 2$, Dufort-Frankel is used.

<u>Stability</u> :-

$\Rightarrow$ A scheme is stable if round-off errors are not amplified in the calculations.

It is checked using Von-Neumann analysis (Fourier Method).

Assume the numerical scheme admits a soln

$$u_J^n = a^{(n)}(\omega)\, e^{iJ\omega\Delta x} = a^n e^{iJ\omega\Delta x}$$

where, ω is the wave number and $i = \sqrt{-1}$.

$G(\omega) = \dfrac{a^{(n+1)}(\omega)}{a^{(n)}(\omega)}$ is called an amplification factor, which controls the growth of the Fourier component $a(\omega)$.

Stability condition: $|G(\omega)| \leq 1$. for $0 \leq \omega\Delta x \leq \pi$.

Bender - Schmidt :-

$$u_t - \nu u_{xx} = 0$$

$$\Rightarrow \quad u_j^{n+1} = \lambda u_{j-1}^n + (1-2\lambda) u_j^n + \lambda u_{j+1}^n$$

$$\Rightarrow \quad G(\omega) = \frac{a^{(n+1)}(\omega)}{a^{(n)}(\omega)} = 1 - 4\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right)$$

For stability, $|G(\omega)| \le 1$

$$\Rightarrow \quad 0 \le 4\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right) \le 2$$

$$\Rightarrow 1 - 4\lambda \le G(\omega) \le 1$$

So, $|G(\omega)| \le 1$ if $1 - 4\lambda \ge -1$.

$$\Rightarrow \quad 4\lambda \le 2$$

$$\therefore \quad \lambda \le \frac{1}{2}$$

Crank - Nicholson :

$$G(\omega) = \frac{1 - 2\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right)}{1 + 2\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right)}.$$

$|G(\omega)| \le 1 \quad \forall \lambda > 0$. Unconditionally Stable.

DF : Put $u_j^n = a^n e^{ij\omega\Delta x}$ in Dufort - Frankel scheme to get:

$$(1+\mu) a^2 - 2a\mu \cos(\omega \Delta x) - (1-\mu) = 0$$

$$\Rightarrow \quad a_{\pm} = \frac{\mu \cos(\omega \Delta x) \pm \sqrt{1 - \mu^2 \sin^2(\omega \Delta x)}}{(1+\mu)}$$

Case-I $\quad 1 - \mu^2 \sin^2(\omega \Delta x) \ge 0 \Rightarrow |a_{\pm}| \le \dfrac{|\mu \cos(\omega \Delta x) + 1|}{(1+\mu)} \le 1$.

Case-II $\quad 1 - \mu^2 \sin^2(\omega \Delta x) < 0 \Rightarrow |a_{\pm}|^2 = \dfrac{\mu^2 - 1}{(1+\mu)^2} = \dfrac{\mu - 1}{1+\mu} < 1$

So, for all values of μ, it is stable.

Exm :- Solve $u_t = u_{xx}$ subject to

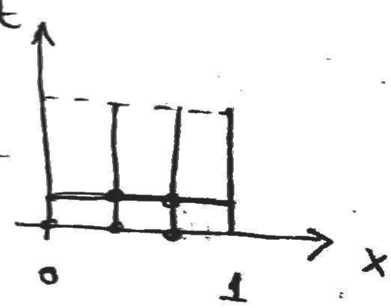
$u(x,0) = \sin(\pi x)$, $0 \le x \le 1$, $u(0,t) = 0 = u(1,t)$

using Schmidt, Crank-Nicolson & Dufort Frankel methods. Take $\Delta x = \frac{1}{3}$, $\Delta t = \frac{1}{36}$.

Soln :- Check $\lambda = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{1/36}{1/9} = \frac{1}{4} < \frac{1}{2}$.

So, Bender-Schmidt method is stable.

S1 $[0,1]$ is divided into 4 nodal points: $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = 2/3$, $x_3 = 1$



Let, $t_n = n\Delta t = \dfrac{n}{36}$, $n \ge 0$.

Assume: $u(x_i, t_n) \simeq u_i^n$. $0 \le i \le 3$, $n \ge 0$.

S2 $u_t - u_{xx} = 0$

BS $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} = \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$, $1 \le i \le 2$, $n \ge 0$

$$u_i^{n+1} = \lambda u_{i-1}^n + (1-2\lambda) u_i^n + \lambda u_{i+1}^n$$

i.e. $$\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda \\ \lambda & 1-2\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/4 \\ \frac{1}{4} & 1/2 \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \end{pmatrix}$$

with $U^0 = \begin{pmatrix} \sin(30) \\ \sin(60) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ \sqrt{3}/2 \end{pmatrix}$

$$U^1 = \begin{pmatrix} 0.65 \\ 0.65 \end{pmatrix}$$

CN

$$u_t - u_{xx} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right]$$

$$\lambda = \frac{\Delta t}{\Delta x^2} = \frac{1}{4}$$

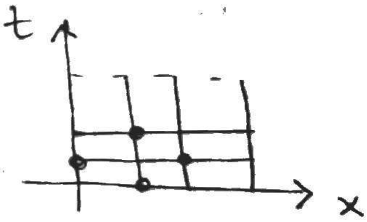
So, $$(1+\lambda)u_i^{n+1} - \frac{\lambda}{2}u_{i-1}^{n+1} - \frac{\lambda}{2}u_{i+1}^{n+1} = \frac{\lambda}{2}u_{i-1}^n + (1-\lambda)u_i^n + \frac{\lambda}{2}u_{i+1}^n$$

$i = 1,2, \ n \geq 0$

$$\begin{pmatrix} 1+\lambda & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1+\lambda \end{pmatrix}\begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\lambda & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1-\lambda \end{pmatrix}\begin{pmatrix} u_1^n \\ u_2^n \end{pmatrix}$$

i.e. $$\begin{pmatrix} 5/4 & -1/8 \\ -1/8 & 5/4 \end{pmatrix} U^{n+1} = \begin{pmatrix} 3/4 & 1/8 \\ 1/8 & 3/4 \end{pmatrix} U^n$$ with $(n \geq 0)$

$$U^0 = \begin{pmatrix} \sqrt{3}/2 \\ \sqrt{3}/2 \end{pmatrix}$$



Then, $$\underline{U^1 = \begin{pmatrix} 0.67 \\ 0.67 \end{pmatrix}}$$

DF

$$u_t - u_{xx} = 0$$

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{u_{i+1}^n - u_i^{n-1} - u_i^{n+1} + u_{i-1}^n}{\Delta x^2}$$

$$\Rightarrow (1+\mu)u_i^{n+1} = (1-\mu)u_i^{n-1} + \mu(u_{i-1}^n + u_{i+1}^n), \quad \underline{\mu = 2\lambda}.$$

$i = 1,2, \ n \geq 1$

Given U^0, and calculated U^1 (by CN)

$$U^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \end{pmatrix} = \left(\begin{array}{cc|cc} \frac{1-\mu}{1+\mu} & 0 & 0 & \frac{\mu}{1+\mu} \\ 0 & \frac{1-\mu}{1+\mu} & \frac{\mu}{1+\mu} & 0 \end{array}\right) \left(\begin{array}{c} u_1^{n-1} \\ u_2^{n-1} \\ \hline u_1^{n} \\ u_2^{n} \end{array}\right)$$

$$\therefore \quad U^{n+1} = \left(\begin{array}{cc|cc} 1/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \end{array}\right) \left(\begin{array}{c} U^{n-1} \\ \hline U^{n} \end{array}\right)$$