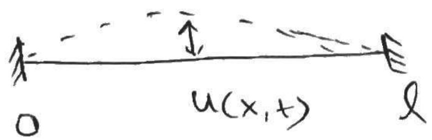
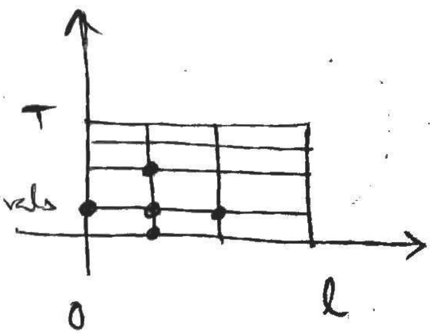
Hyperbolic PDE :-

$$\begin{cases} u_{tt} - c^2 u_{xx} = \bar{f}(x,t) & x \in [0,l],\ t>0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \\ u(0,t) = g_1(t), \quad u(l,t) = g_2(t) \end{cases}$$





S1 : Divide $[0,l]$ into N subintervals

i.e. $\Delta x = \dfrac{l}{N}$

So, $x_i = i\Delta x,\ 0 \leq i \leq N$

and $\Delta t = \dfrac{T}{m}$ $\therefore t_n = n\Delta t,\quad n \geq 0.$

S2 $u_{tt} - c^2 u_{xx} = \bar{f}(x,t)$

Let, $u(x_i, t_n) \simeq u_i^n$

Then, $u_{tt} - c^2 u_{xx} = f(x,t)$

$$\Rightarrow \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} - c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \bar{f}_i^n$$

Let, $r = c^2 \frac{\Delta t^2}{\Delta x^2}$

Then,

$$u_i^{n+1} = r\left(u_{i-1}^n + u_{i+1}^n\right) + 2(1-r)u_i^n - u_i^{n-1} + \bar{f}_i^n . \Delta t^2 \quad \cdots (1)$$

$$1 \leq i \leq N-1, \quad n \geq 0.$$

In matrix form:

$$U^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{pmatrix} = \begin{pmatrix} 2(1-r) & r & 0 & \cdots & 0 \\ r & 2(1-r) & r & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & & r & 2(1-r) \end{pmatrix} \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-1}^n \end{pmatrix}$$

$$- \begin{pmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ u_{N-1}^{n-1} \end{pmatrix} + \begin{pmatrix} \Delta t^2 \bar{f}_1^n + r u_0^n \\ \Delta t^2 \bar{f}_2^n \\ \vdots \\ \Delta t^2 \bar{f}_{N-2}^n \\ \Delta t^2 \bar{f}_{N-1}^n + r u_N^n \end{pmatrix}$$

$$\therefore \underline{U^{n+1} = A U^n - U^{n-1} + b^n.} \quad , n \geq 0.$$

with $u_0^n = g_1(n\Delta t)$, $u_N^n = g_2(n\Delta t)$

$\underline{U^0 = [f(x_1) \;\; f(x_2) \; \cdots \; f(x_{N-1})]^T}$. How to get $\underline{U^{-1}}$?

$u_t(x, 0) = g(x)$

So, $\frac{u_i^1 - u_i^{-1}}{2\Delta t} = g(x_i) \Rightarrow \underline{u_i^{-1} = u_i^1 - 2\Delta t\, g_i} \quad \underline{1 \leq i \leq N-1}$.

Also, (1) : $\underline{\underline{n=0}}$

$$u_i^1 = r\left(u_{i-1}^0 + u_{i+1}^0\right) + 2(1-r)u_i^0 - u_i^{-1} + \bar{f}_i^0 . \Delta t^2.$$

$$\therefore u_i^1 = \frac{1}{2} r\left(u_{i-1}^0 + u_{i+1}^0\right) + (1-r)u_i^0 + \cancel{2}\Delta t\, g_i + \frac{\Delta t^2}{2} \bar{f}_i^0.$$

$\underline{1 \leq i \leq N-1}$

So,

$$U^1 = \begin{pmatrix} 1-r & \frac{r}{2} & 0 & \cdots & 0 \\ \frac{r}{2} & 1-r & \frac{r}{2} & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & & \frac{r}{2} & 1-r \end{pmatrix} U^0 + \begin{pmatrix} \Delta t\, g_1 + \frac{\Delta t^2}{2} \bar{f}_1^0 + \frac{r}{2} u_0^0 \\ \Delta t\, g_2 + \frac{\Delta t^2}{2} f_2^0 \\ \vdots \\ \Delta t\, g_{N-1} + \frac{\Delta t^2}{2} f_{N-1}^0 + \frac{r}{2} u_N^0 \end{pmatrix}$$

Algorithm :-

1. $$U^1 = \frac{A}{2} U^0 + c^0$$

2. $$U^{n+1} = A U^n - U^{n-1} + b^n \quad , \quad n \geq 1$$

By Von-Neumann Analysis, put $u_j^n = a^n e^{ij\omega\Delta x}$ to prove that the explicit method is stable for $r \leq 1$.

EXM :- $u_{tt} - 4u_{xx} = 0$, $u(x,0) = x(4-x)$, $u_t(x,0) = 0$ and $u(0,t) = 0 = u(4,t)$.

Soln :-

S1 Didive $[0,4]$ into 4 sub-intervals, i.e.

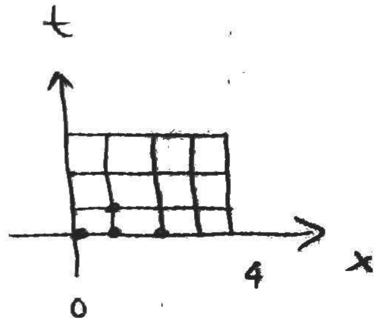
$$\Delta x = \frac{4}{4} = 1$$

$$r = c^2 \frac{\Delta t^2}{\Delta x^2} = 4 \frac{\Delta t^2}{1} \Rightarrow \Delta t \leq \frac{1}{2}$$

Let, $\Delta t = \frac{1}{2}$.

Then, $x_i = i$, $0 \leq i \leq 4$

$t_n = \frac{n}{2}$, $n \geq 0$.



S2 $u_t(x,0) = 0$

$$\Rightarrow \frac{u_i^1 - u_i^{-1}}{2\Delta t} = 0$$

$$\Rightarrow u_i^1 = u_i^{-1} \quad , \quad 1 \leq i \leq 3$$

$u_{tt} - 4u_{xx} = 0$

$$\Rightarrow \frac{u_i^1 - 2u_i^0 + u_i^{-1}}{\Delta t^2} = 4 \frac{u_{i+1}^0 - 2u_i^0 + u_{i-1}^0}{\Delta x^2}$$

$$\Rightarrow \quad 2u_i^1 = u_{i+1}^0 + u_{i-1}^0$$

$$\therefore \quad u_i^1 = \frac{1}{2}\left(u_{i+1}^0 + u_{i-1}^0\right) \qquad 1 \le i \le 3.$$

$$U^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{pmatrix} = \frac{A}{2} U^0$$

with $\quad U^0 = [3 \quad 4 \quad 3]^T.$

Also, $\quad u_{tt} - 4u_{xx} = 0$

$$\Rightarrow \quad \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = 4 \, \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\Rightarrow \quad u_i^{n+1} = \left(u_{i-1}^n + u_{i+1}^n\right) \qquad 1 \le i \le 3, \quad \underline{n \ge 1}.$$

$$\therefore \quad U^{n+1} = A U^n, \qquad \underline{n \ge 1}.$$

with $\quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and

$U^0 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \quad U^1 = \frac{A}{2} U^0.$