(1)

Fundamental Theorem of calculus :-

I) Let, f is continuous on $[a,b]$, and F is defined by

$$F(x) = \int_a^x f(t)\,dt \,.$$

Then, F is continuous on $[a,b]$ and differentiable on (a,b) and

$$F'(x) = f(x) \,, \quad x \in (a,b)$$

(belongs to)

II) Let, f & F are two functions in $[a,b]$ ~~as~~. such that

$$F'(x) = f(x) \,, \quad x \in (a,b)$$

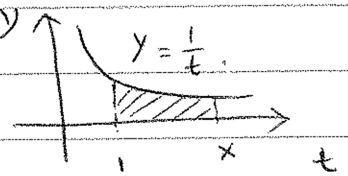
Then, $\int_a^b f(x)\,dx = F(b) - F(a)$.

(2)

Natural Logarithm

Defn : $\ln(x) = \int_1^x \frac{1}{t}\, dt, \quad x > 0$

$\Rightarrow$ The fn exists as $\frac{1}{x}$ is continu on $(0,\infty)$. And $\ln(x)$ is cont.



For $x > 1$, then $\ln(x)$ is the area under $y = \frac{1}{t}$.

$\Rightarrow \quad \ln(1) = \int_1^1 \frac{1}{t}\, dt = 0.$

For $0 < x < 1$, $\quad \ln(x) = \int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt < 0$

By Fundamental theorem of calculus ~~(2nd)~~ (1st)

$$\frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}.$$

$$\Rightarrow \frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

Prop : i) $\ln(xy) = \ln(x) + \ln(y)$

ii) $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

iii) $\ln(x^r) = r\ln(x)$, r = rational

iv) $\lim_{x\to\infty} \ln(x) = \infty$

v) $\lim_{x\to 0+} \ln(x) = -\infty$

exponential.

~~$\ln$ is 1-1. as $\frac{d}{dx}\ln(x) = \frac{1}{x} > 0$~~

Defn ~~$\ln : (0,\infty) \to (-\infty,\infty)$~~

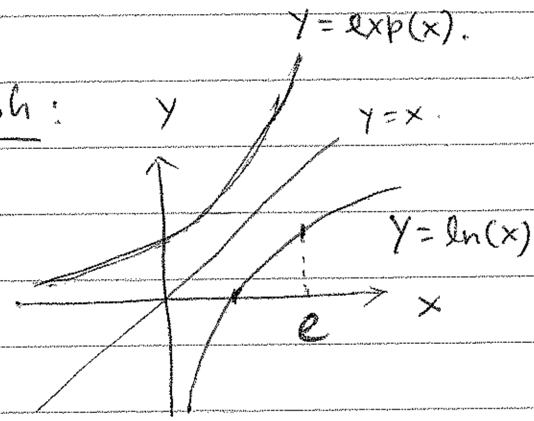
Define $\exp : (-\infty,\infty) \to (0,\infty)$ as the inverse fn of $\ln$.

i.e.

$$\exp(x) = y \Leftrightarrow \ln(y) = x.$$

$\Rightarrow \quad \exp(0) = 1$

$\Rightarrow$ Graph :



$\Rightarrow \quad \ln(e) := 1$, ($e$ is a number.)

$e = 2.718\ldots$

$\Rightarrow \quad \exp(1) = e.$

$\ln(e^r) = r\ln(e) = r$

$\Rightarrow \quad \exp(r) := e^r$ for r rational

So, $\exp(x) = e^x$, x real.

$\Rightarrow$

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N-log

i) $\frac{d}{dx}\left(\ln(u)\right) = \frac{1}{u}\frac{du}{dx}$.

ii) $\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$.

(Chain rule).

exm: $\frac{d}{dx}\ln\left(\frac{x^2+1}{\sqrt{x^2+2}}\right)$

Soln: $F(x) = \ln\left(\frac{x^2+1}{\sqrt{x^2+2}}\right)$

$= \ln(x^2+1) - \ln(\sqrt{x^2+2})$

$= \ln(x^2+1) - \frac{1}{2}\ln(x^2+2)$

$\frac{d}{dx}F(x) = \frac{\frac{d}{dx}(x^2+1)}{x^2+1} - \frac{1}{2}\frac{\frac{d}{dx}(x^2+2)}{x^2+2}$

$= \frac{2x}{x^2+1} - \frac{x}{x^2+2}$

$= \frac{2x^3+4x-x^3-x}{(x^2+1)(x^2+2)}$

$= \frac{x^3+3x}{(x^2+1)(x^2+2)}$

Find $f'(x)$ if $f(x) = \ln|x|$.

Soln

$f(x) = \begin{cases} \ln(x), & x>0 \\ \ln(-x), & x<0 \end{cases}$

exponential.

$e^x = \exp(x)$, for x real.

$e^x = y \Leftrightarrow \ln(y) = x$.

Cancellation eqn.

$e^{\ln(x)} = x$, $x>0$

& $\ln(e^x) = x$. $\forall x$.

Prop.

i) $e^x \cdot e^y = e^{x+y}$

ii) $e^{x-y} = e^x/e^y$.

iii) $(e^x)^r = e^{xr}$.

iv) $\lim_{x\to-\infty} e^x = 0$

v) $\lim_{x\to\infty} e^x = \infty$.

$\frac{d}{dx}(e^x) = e^x$.

Let, $y = e^x$

$\Rightarrow \ln y = x$.

$\frac{d}{dx}(\ln(y)) = 1$

$f(x) = \ln(x)$

$\frac{d}{dx}(e^x)$

$= (f^{-1})'(x)$

$= \frac{1}{f'(f^{-1}(x))}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y = e^x$.

Rule: $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$.

So, $\int e^x dx = e^x + c$

④

So, $f'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{+1}{+x}, & x < 0 \end{cases}$

$\therefore \quad f'(x) = \frac{1}{x}, \quad x \neq 0$

$$\frac{d}{dx}\left(\ln(|x|)\right) = \frac{1}{x}.$$

$$\Rightarrow \quad \int \frac{1}{x}\,dx = \ln(|x|) + c.$$

⑩ Applications Integration & Diff.

* Find $I = \int_e^6 \frac{dx}{x \ln x}$.

Take $\ln(x) = u$

$\Rightarrow \frac{1}{x}dx = du$.

Soln:

$$I = \int_1^{\ln(6)} \frac{du}{u}$$

$$= \ln(|u|)\Big]_1^{\ln(6)} = \boxed{\ln(\ln(6))}.$$

63 $y = \sqrt{\frac{x-1}{x^4+1}}$ Find $\frac{dy}{dx}$

$$\ln(y) = \ln\sqrt{\frac{x-1}{x^4+1}} = \frac{1}{2}\ln\frac{x-1}{x^4+1}$$

$$= \frac{1}{2}\left(\ln(x-1) - \ln(x^4+1)\right)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\,\frac{4x^3}{x^4+1}$$

$$\therefore \frac{dy}{dx} = \frac{y}{2}\left(\frac{x^4+1-4x^4+4x^3}{(x-1)(x^4+1)}\right)$$

$$= \frac{1}{2}\sqrt{\frac{x-1}{x^4+1}}\;\frac{4x^3-3x^4+1}{(x-1)(x^4+1)}$$

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52 $f(t) = \sin^2\left(e^{\sin^2 t}\right)$ Find $f'(t)$.

$\cancel{\frac{d}{dt} f(t) = 2\sin\left(e^{\sin^2 t}\right)}$

Let, $e^{\sin^2 t} = u$

$f(t) = \sin^2 u$

$\therefore \frac{df}{dt} = 2\sin u \cdot \cos u \cdot \frac{du}{dt} \cdot \quad \left(\frac{df}{du} \cdot \frac{du}{dt}\right).$

$= \sin 2u \cdot e^{\sin^2 t} \cdot 2\sin t \cdot \cos t.$

$= e^{\sin^2 t} \sin\left(2e^{\sin^2 t}\right) \sin 2t.$

90. $\int_0^1 e^x \cos(e^x)\, dx.$ Let, $u = e^x.$

$du = e^x\, dx.$

$= \int_1^e \cos u\, du. = \sin u \Big]_1^e$

$= \sin(e) - \sin(1).$

Q. H.W : Find $f'(x)$, $f(x) = \int_2^{e^x} \ln(t\sin t)\, dt.$

Hint: $h(x) = \int_2^x \ln(t\sin t)\, dt.$

$h'(x) = \ln(x\sin x)$

$\frac{d}{dx}[h(e^x)] = h'(e^x) \cdot \frac{d}{dx}(e^x)$

Ex Find the slope of the tangent to the curve $y = x^{\sin x}$ at $x = \pi$.

$$\ln(y) = (\sin x) \cdot \ln(x)$$

$$\Rightarrow \frac{1}{y} y' = \cos x \ \ln(x) + \frac{1}{x} \sin x$$

$$\therefore \ y' = y \left(\cos x \cdot \ln(x) + \frac{\sin x}{x} \right)$$

$$y' \Big|_{x=\pi} = \pi^0 \left((-1) \cdot \ln(\pi) \right)$$

$$= \boxed{-\ln(\pi)}$$