Iterative Methods :-

Gauss elimination or LU decomposition techniques are direct methods, i.e. a solution is obtained after a single application of GE or LU. Once a solution is obtained, GE offers no method of refinement.

⇒ Iterative methods begin with an initial guess x_0, then obtain a series of improved approximations $x_1, x_2, \ldots$ that converge to the exact solution.

⇒ This method can be stopped as soon as the approx. x_n have converged to an acceptable precision. In direct method, bailing out early, is not an option.

For large and sparse matrices, iterative methods are faster.

Jacobi Method :-

Exm :

$$7x_1 - x_2 = 6$$
$$x_1 - 5x_2 = -4$$

We rewrite the system as:

$$\left.\begin{aligned} x_1 &= \frac{1}{7}(6 + x_2) \\ \& \quad x_2 &= \frac{1}{5}(4 + x_1) \end{aligned}\right\} \cdots (1)$$

Then we iterate the approx. as:

$$x_1^{(k)} = \frac{1}{7}\left(6 + x_2^{(k-1)}\right)$$
$$x_2^{(k)} = \frac{1}{5}\left(4 + x_1^{(k-1)}\right)$$

for $k = 1, 2, \ldots$ with an initial approx $x_1^{(0)} = 0$, $x_2^{(0)} = 0$.

k	0	1	2	3	4
$x_1^{(k)}$	0	$6/7$	$34/35$	0.9959	0.9994
$x_2^{(k)}$	0	$4/5$	$34/35$	0.9943	0.9992

Clearly $x_1^{(k)} \to 1$, $x_2^{(k)} \to 1$, the exact solⁿ.

Gauss-Seidel Method :-

One can modify Jacobi iteration to get faster (hopefully) convergence.

We iterate ① as follows:

$$x_1^{(k)} = \frac{1}{7}\left(6 + x_2^{(k-1)}\right)$$ and then

use the improved approx $x_1^{(k)}$ to calculate:

$$x_2^{(k)} = \cancel{4}\ \frac{1}{5}\left(4 + x_1^{(k)}\right)$$ with an initial approx. $x_2^{(0)} = 0$. ($x_1^{(0)}$ does not matter)

Then,
$$x_1^{(1)} = 6/7, \quad x_2^{(1)} = \frac{34}{35}$$
$$x_1^{(2)} = 0.9959, \quad x_2^{(2)} = 0.9992$$

k	0	1	2
$x_1^{(k)}$	0	$6/7$	0.9959
$x_2^{(k)}$	0	$34/35$	0.9992

Now try the system:

$$\left.\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6. \end{aligned}\right\}$$

So that
$$x_1 = 5x_2 - 4$$
$$x_2 = 7x_1 - 6$$

Thus Jacobi :
$$x_1^{(k)} = 5x_2^{(k-1)} - 4$$
$$x_2^{(k)} = 7x_1^{(k-1)} - 6.$$

or GS:

$$x_1^{(k)} = 5x_2^{(k-1)} - 4$$

$$x_2^{(k)} = 7x_1^{(k)} - 6$$

The table becomes: Jacobi

k	0	1	2	3	4
$x_1^{(k)}$	0	-4	-34	-174	-1244
$x_2^{(k)}$	0	-6	-34	-244	-1244

or <u>GS</u>

k	0	1	2
$x_1^{(k)}$	0	-4	-174
$x_2^{(k)}$	0	-34	-1224

i.e. both Jacobi & GS diverge.

<u>Definition</u> :- (<u>Diagonally</u> dominant)

An $n \times n$ matrix A is strictly diagonally dominant if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row.

i.e.

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for } i = 1, 2, \dots, n.$$

<u>Result</u> :- If A is strictly diagonally dominant, then the system $Ax = b$ has a unique solution to which the Jacobi and the Gauss-Seidel method will converge for any initial approximation.

(Mises, Pollaczek - Geiringer)

$$Ax = b \equiv \begin{pmatrix} 7 & -1 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

A is diag dominant, but

$$\begin{pmatrix} 1 & -5 \\ 7 & -1 \end{pmatrix} \text{ is not diag. dom.}$$

In matrix notation:-

$$A = L + D + U. \quad \text{with } D = \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Jacobi Method:-

$$Dx = -(L+U)x + b$$

In iteration,

$$Dx^{(k)} = -(L+U)x^{(k-1)} + b$$

i.e. $x^{(k)} = -D^{-1}(L+U)x^{(k-1)} + D^{-1}b$

with initial guess $x^{(0)}$.

Gauss-Seidel Method:-

$$(D+L)x = -Ux + b$$

In iteration,

$$x^{(k)} = -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b$$

with initial guess $x^{(0)}$.

Algebraically,

Jacobi: $x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}\right)$. $i = 1, 2, \ldots, n$

&

GS: $x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)}\right)$

Some observations :-

Jacobi :

$$x^{(k)} = -D^{-1}(L+U)\, x^{(k-1)} + D^{-1}b$$
$$= M_J\, x^{(k-1)} + D^{-1}b \quad , \text{ say.}$$

M_J is the Jacobi iteration matrix.

$$M_J = -D^{-1}(L+U) = D^{-1}(D-A)$$
$$= I - D^{-1}A$$

$$\therefore \quad I - M_J = D^{-1}A$$

$\Rightarrow$ A diag. dominant $\Rightarrow \|I - D^{-1}A\|_{L^\infty} = \|M_J\|_{L^\infty} = \max_i \left|\dfrac{\sum_{j\neq i} a_{ij}}{a_{ii}}\right| < 1$.

Proof

$\overset{?}{\Rightarrow}$ $D^{-1}A$ is invertible $\Rightarrow$ A is invertible.

Let, $D^{-1}A$ is singular, then $\exists\, x \neq 0$ s.t

$$D^{-1}Ax = 0$$

$0 \neq \|x\| = \|(I - D^{-1}A)x\|_\infty \leq \|I - D^{-1}A\|_\infty \|x\|_\infty < \|x\|_\infty$

which is absurd.

Gauss-Seidel :-

$$x^{(k)} = -(D+L)^{-1}U\, x^{(k-1)} + (D+L)^{-1}b$$
$$= M_G\, x^{(k-1)} + (D+L)^{-1}b.$$

M_G is the Gauss-Seidel iteration matrix.

$$M_G = -(D+L)^{-1}U$$
$$= -(D+L)^{-1}(A-(L+D))$$
$$= I - (L+D)^{-1}A$$

Exm:-

$$4x_1 + x_2 + x_3 = 2$$
$$x_1 + 5x_2 + 2x_3 = -6$$
$$x_1 + 2x_2 + 3x_3 = -4$$

Exact soln: $= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

By Jacobi & Gauss-Seidel. (in matrix form).

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Jacobi's iteration :—

$$D x^{(k)} = -(A - D)\, x^{(k-1)} + b$$

$$\Rightarrow \quad x^{(k)} = (I - D^{-1}A)\, x^{(k-1)} + D^{-1} b$$

$$D^{-1}A = \begin{pmatrix} 1/4 & & 0 \\ & 1/5 & \\ 0 & & 1/3 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1/4 & \frac{1}{4} \\ \frac{1}{5} & 1 & 2/5 \\ \frac{1}{3} & 2/3 & 1 \end{pmatrix}$$

So,

$$x^{(k)} = \begin{pmatrix} 0 & -1/4 & -1/4 \\ -\frac{1}{5} & 0 & -2/5 \\ -\frac{1}{3} & -2/3 & 0 \end{pmatrix} x^{(k-1)} + \begin{pmatrix} 2/4 \\ -6/5 \\ -4/3 \end{pmatrix}.$$

With zero initial guess,

$$x^{(1)} = \begin{pmatrix} 2/4 \\ -6/5 \\ -4/3 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 1.133 \\ -0.767 \\ -0.700 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 0.867 \\ -1.147 \\ -1.200 \end{pmatrix}$$

$$x^{(4)} = \begin{pmatrix} 1.087 \\ -0.893 \\ -0.858 \end{pmatrix}$$

Gauss Seidel iteration:

$$(L+D)x^{(k)} = -Ux^{(k-1)} + b$$

$$\Rightarrow \quad x^{(k)} = -(L+D)^{-1}U x^{(k-1)} + (L+D)^{-1} b.$$

$$D+L = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 2 & 3 \end{pmatrix}, \quad (D+L)^{-1}U = \begin{pmatrix} 1/4 & 0 & 0 \\ -1/20 & 1/5 & 0 \\ -1/20 & -2/15 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/4 & 1/4 \\ 0 & -1/20 & 7/20 \\ 0 & -1/20 & -19/60 \end{pmatrix}$$

and $$(D+L)^{-1} b = \begin{pmatrix} 1/4 & 0 & 0 \\ -1/20 & 1/5 & 0 \\ -1/20 & -2/15 & 1/3 \end{pmatrix} \begin{pmatrix} 2 \\ -6 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -13/10 \\ -19/30 \end{pmatrix}$$

So, $$x^{(k)} = \begin{pmatrix} 0 & -1/4 & -1/4 \\ 0 & 1/20 & -7/20 \\ 0 & 1/20 & 19/60 \end{pmatrix} x^{(k-1)} + \begin{pmatrix} 1/2 \\ -13/10 \\ -19/30 \end{pmatrix}.$$

With zero initial guess, the iteration will be:

$$x^{(1)} = \begin{pmatrix} 1/2 \\ -13/10 \\ -19/30 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 0.983 \\ -1.143 \\ -0.899 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 1.011 \\ -1.043 \\ -0.975 \end{pmatrix}$$

$$x^{(4)} = \begin{pmatrix} 1.004 \\ -1.011 \\ -0.994 \end{pmatrix}.$$

If only diag. dom. (not strict)

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Both Jacobi & G.S iterative schemes can be written as:

$$x^{(k)} = P x^{(k-1)} + q \quad \cdots (*)$$

where P is a constant matrix and q is a constant vector.

Jacobi: $P = M_J = I - D^{-1}A$

GS: $P = M_G = I - (L+D)^{-1}A$

<u>Convergence</u>:- (Nekrasov / Pizetti) The iterative scheme $(*)$ is convergent iff every eigen-value of P satisfies $|\lambda| < 1$, i.e. the spectral radius $\rho(P) < 1$.

$\Rightarrow$ If $\|P\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} < 1$ for some norm, then $(*)$ will converge.

$$x^* = Px^* + q$$

$$\Rightarrow \quad x^* - x^{(k)} = P(x^* - x^{(k-1)})$$

$$\Rightarrow \quad \underline{e_k = P e_{k-1}} \quad \Rightarrow \|e_k\| \leq \|P\| \, \|e_{k-1}\|$$

$\Rightarrow$ Jacobi: ~~G.S~~:

$$P = -D^{-1}(L+U)$$

$$= -\begin{pmatrix} 1/a_{11} & & 0 \\ & \ddots & \\ 0 & & 1/a_{nn} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & \cdot & a_{1n} \\ \vdots & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & \cdots & 0 \end{pmatrix}$$

A=diagonally dominant

$$= \begin{pmatrix} 0 & -a_{12}/a_{11} & \cdots & -a_{1n}/a_{11} \\ \vdots & \ddots & \ddots & \\ \vdots & & & \\ -a_{n1}/a_{nn} & & & 0 \end{pmatrix} \Rightarrow \sum_{j=1}^{n} |P_{ij}| < 1 \ \forall i$$

$$\Rightarrow \|P\|_\infty < 1.$$

Thm If $n=2$, Jacobi iteration converges iff GS converges.

Proof:

$$\rho(M_J) = \sqrt{\left|\frac{a_{21}a_{12}}{a_{11}a_{22}}\right|}$$

$$\& \quad \rho(M_{GS}) = \frac{|a_{21}a_{12}|}{|a_{11}a_{22}|}$$

$$\rho(M_J) < 1 \quad \text{iff} \quad \rho(M_{GS}) < 1$$

Thm For any $n>2$, it is possible for Jacobi iteration to converge while GS diverges and conversely.

$$Ax = b$$

$$A = B - C$$

$$Bx^{(k)} = Cx^{(k-1)} + b$$

$$\Rightarrow \quad x^{(k)} = (B^{-1}C)\,x^{(k-1)} + B^{-1}b$$

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & & a \\ 1 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ b & & & & 1 & 1 \end{pmatrix} \quad \text{for } a, b \in \mathbb{R}.$$

For Jacobi, $\det(\lambda I - M_J) = 0$ gives

$$\lambda^n - ab\lambda^{n-2} + (-1)^{n+1}a = 0$$

For GS, we get

$$\lambda^{n-1}\left(\lambda - a\left(b + (-1)^n\right)\right) = 0$$

i) $a \geq 1,\ b = (-1)^{n+1} \Rightarrow \rho(M_{GS}) = 0,\ \rho(M_J) \geq 1$

ii) $a = \frac{1}{2}(-1)^{n+1},\ b = (-1)^{n} \Rightarrow \rho(M_{GS}) = 1$ and $\rho(M_J) < 1$

<u>Exm</u>

$$A_J = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad A_J = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$\boxed{\rho(M_J) = 0.84 \quad \rho(M_{GS}) = 1}$

$\boxed{\rho(M_J) = 0.92 \quad \rho(M_{GS}) = 1}$

$$\# \quad A_J = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

$\boxed{\rho(M_J) = 0.71,\ \rho(M_{GS}) = 0.5}$

J conv.

GS div

$n > 2$

<u>Exm</u>

$$A_{GS} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad A_{GS} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$\boxed{\rho(M_J) = 1.27 \quad \rho(M_{GS}) = 0}$

$\boxed{\rho(M_J) = 1.32 \quad \rho(M_{GS}) = 0}$

$$\# \quad A_{GS} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

J div

GS conv.

$n \geq 2$

$\boxed{\rho(M_J) = 1;\ \rho(M_{GS}) = 1}$

<u>Jacobi Algorithm</u>:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)} \right], \quad i = 1, 2, \ldots, n$$

<u>Gauss - Seidel Algorithm</u>:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

$i = 1, 2, \ldots, n.$

Theorem 2. Let, $\| \ \|$ be a matrix norm in $M_n(\mathbb{C})$. It satisfies $\rho(A) \leq \|A\|$.

Proof: Let, $\rho(A) = |\lambda|$ and $x \neq 0$ is the corresponding eigen vector

$$\|\lambda x\| = \rho(A)\|x\| = \|Ax\| \leq \|A\|\|x\|$$ if $\|\cdot\|$ is an operator norm / induced norm

$$\Rightarrow \quad \underline{\rho(A) \leq \|A\|}$$

If $\|\cdot\|$ is not an induced norm, but a consistent norm, then take $y \neq 0 \in M_n(\mathbb{C})$ such that

$$xy^T \neq 0$$

$$Axy^T = \lambda xy^T.$$

Now, $\|\lambda xy^T\| = \|Axy^T\| \leq \|A\|\|xy^T\|$

$\quad \| $

$|\lambda| \|xy^T\|$

$$\Rightarrow \quad \underline{\rho(A) \leq \|A\|}.$$

Note: J & GS converse iff $\rho(A) < 1$.

So, if for some matrix norm $\|\cdot\|$, $\|A\| < 1$ then $\rho(A) \leq \|A\| < 1$. so that J & GS. converse.

Theorem 1. Let, $\| \, \|$ be an induced matrix norm. If $\|P\| < 1$, then $x^{(k+1)} = P x^{(k)} + c$ converges for any $x^{(0)}$

Proof let, $e_{k+1} = x^{(k+1)} - x$ is the error.

Then $x^{(k+1)} = P x^{(k)} + c$ & $x = Px + c$ give

$$e_{k+1} = P e_k.$$

$$\Rightarrow \| e_{k+1} \| \le \|P\| \, \|e_k\|$$

$$\le \|P\|^{k+1} \|e_0\|$$ which converges to 0 as $k \to \infty$ if $\|P\| < 1$

Theorem 3. For all P and $\epsilon > 0$ $\exists$ an induced norm $\| \cdot \|_I$ such that $\|P\|_I \le \rho(P) + \epsilon$.

Hint: $\exists$ S such that

$S^{-1} P S = J$, a Jordan form.

Let, $D_\epsilon = \text{diag}(1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1})$. Then

$$(SD_\epsilon)^{-1} P (SD_\epsilon) = D_\epsilon^{-1} J D_\epsilon$$

$$= \left(\begin{array}{ccc|cc|c} \lambda_1 & \epsilon & & & & \\ & \lambda_1 & \epsilon & & & \\ & & \lambda_1 \ddots & & & \\ \hline & & & \lambda_2 & \epsilon & \\ & & & & \lambda_2 \ddots & \\ \hline & & & & & \end{array} \right)$$

Define the vector norm:

$$\|x\|_I = \|(SD_\epsilon)^{-1} x\|_\infty$$ to generate

the induced norm:

$$\|P\|_I = \max_{x \neq 0} \frac{\|Px\|_I}{\|x\|_I}$$

$$= \max_{x \neq 0} \frac{\|(SD_\epsilon)^{-1} Px\|_\infty}{\|(SD_\epsilon)^{-1} x\|_\infty} \qquad [\text{let, } y = (SD_\epsilon)^{-1} x]$$

$$= \max_{y \neq 0} \frac{\|(SD_\epsilon)^{-1} P (SD_\epsilon) y\|_\infty}{\|y\|_\infty}$$

$$= \|(SD_\epsilon)^{-1} P (SD_\epsilon)\|_\infty$$

$$= \max_i |\lambda_i| + \epsilon$$

$$= \rho(P) + \epsilon$$

Theorem 4. The iteration $x^{(k+1)} = Px^{(k)} + c$ converges for any initial vector $x^{(0)}$ iff $\rho(P) < 1$.

Proof:- let, the iteration $x^{(k+1)} \to x$ as $k \to \infty$

If possible, let, $\rho(P) \geq 1$. and $\rho(P) = |\lambda|$

Let, $x^{(0)} - x$ be an eigenvector corresponding to λ, i.e. choose $x^{(0)} = v + x$, v being an eigenvector

Then,

$$x^{(k+1)} - x = P(x^{(k)} - x) = P^{k+1}(x^{(0)} - x)$$

$$= \lambda^{k+1}(x^{(0)} - x) \not\to 0$$

as $k \to \infty$ as $|\lambda| \geq 1$,

a contradiction. So, $\rho(P) < 1$.

Let, $\rho(P) < 1$. There exists an $\epsilon_0 > 0$ s.t

$$\rho(P) + \epsilon_0 < 1$$

By Thm 3, $\exists$ $\|P\|_I$ s.t

$$\|P\|_I \leq \rho(P) + \epsilon_0 < 1$$

So, by Thm 1, $x^{(k+1)} = P x^{(k)} + c$ converges

Defn: Rate of convergence: $r(P) = -\log_{10} \rho(P)$

The smaller the $\rho(P)$, the higher is the rate of convergence.

Successive Over-relaxation (SOR):

To improve GS, we take an appropriate weighted average of $x_i^{(k+1)}$ & $x_i^{(k)}$ to get new $x_i^{(k+1)}$.

G.S: $$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right.$$

SOR: $$\bar{x}_i^{(k+1)} = (1-\omega) x_i^{(k)} + \omega\, x_i^{(k+1)}, \quad \omega \in \mathbb{R}$$

We call it $x_i^{(k+1)}$, itself.

$$x_i^{(k+1)} = (1-\omega)\, x_i^{(k)} + \frac{\omega}{a_{ii}} \left[b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right.$$

In matrix form,

$$(D+\omega L)x^{(k+1)} = \omega b - (\omega U + (\omega-1)D)x^{(k)}$$

$$\therefore \quad x^{(k+1)} = -(D+\omega L)^{-1}(\omega U + (\omega-1)D)\,x^{(k)} + (D+\omega L)^{-1}\omega b$$

$$P_{SOR} = -(D+\omega L)^{-1}(\omega U + (\omega-1)D)$$

If $\omega = 1$: SOR $\equiv$ GS.

If we vary ω, $\rho(P_{SOR})$ smallest may be possible.

In general $\omega_{opt} > 1$: so over relaxation.

$\omega < 1$: under relaxation.

Theorem 5. $\rho(P_{SOR}) \geq |\omega - 1|$

Hence convergence is possible if $\omega \in (0,2)$.

Proof:

$$P_{SOR} = -(D+\omega L)^{-1}(\omega U + (\omega-1)D)$$

$$= \underbrace{(I + \omega D^{-1}L)^{-1}}_{\text{unit lower triangular matrix}}\underbrace{((1-\omega)I - \omega D^{-1}U)}_{\text{upper triangular matrix with diagonal } (1-\omega)}$$

$$\therefore \quad \det(P_{SOR}) = (1-\omega)^n$$

and $\quad |\det(P_{SOR})| \leq (\rho(P_{SOR}))^n$

$$\therefore \quad |1-\omega|^n \leq (\rho(P_{SOR}))^n \Rightarrow \rho(P_{SOR}) \geq |\omega - 1|$$

SOR converges only if $|\omega - 1| < 1$

$$\Rightarrow \underline{0 < \omega < 2}$$

Theorem 6. Let, A is spd. Then SOR converges for $\omega \in (0,2)$.

Proof:

$$P_{SOR} = (D+\omega L)^{-1}(D - \omega D - \omega U)$$

$$= (D+\omega L)^{-1}(D + \omega L - \omega A)$$

$$= I - \omega (D+\omega L)^{-1} A$$

$$= I - \left(L + \frac{1}{\omega} D\right)^{-1} A$$

$$= \underline{I - B^{-1} A}$$

with $B = \left(L + \frac{1}{\omega} D\right)$.

If λ is an eigenvalue of P_{SOR} and x is an eigenvector,

$$P_{SOR}\, x = (I - B^{-1}A)x = \lambda x$$

$$\Rightarrow x - B^{-1}Ax = \lambda x$$

$$\Rightarrow \underline{Ax = (1-\lambda) Bx}$$

Since A is spd, $\lambda \neq 1$. Hence

$$\frac{1}{1-\lambda} = \frac{x^T B x}{x^T A x}$$

Now, $B + B^T = \frac{2}{\omega} D + L + L^T = \left(\frac{2}{\omega} - 1\right) D + A$, A being ~~spd~~ symmetric. So,

$$\mathrm{Re}\left(\frac{1}{1-\lambda}\right) = \frac{1}{2} \frac{x^T(B+B^T)x}{x^T A x} = \frac{1}{2}\left\{\left(\frac{2}{\omega} - 1\right)\frac{x^T D x}{x^T A x} + 1\right.$$

$\therefore \ Re\left(\frac{1}{1-\lambda}\right) > \frac{1}{2}$ if $\omega \in (0,2)$

If $\lambda = u + iv$, we have

$$\frac{1}{2} < Re\left(\frac{1}{1-\lambda}\right) = \frac{1-u}{(1-u)^2+v^2}$$

$$\therefore \ \underline{|\lambda|^2 = u^2 + v^2 < 1}.$$

So, $\underline{\rho(P_{SOR}) < 1}$

Cor: GS converses for spd matrices.

~~(Young, 1950)~~

Thm. ~~If the eigenvalues of $P_J = I - D^{-1}A$ are real and $\rho(P_J) < 1$, then the optimal SOR ω is given by~~

$$\cancel{\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(P_J)^2}}}$$

$$\cancel{\& \ \rho(P_{SOR}) = \omega_{opt} - 1}.$$

Defn: The matrix $A = L + D + U$ is called consistently ordered if the eigen-values of

$$C(\alpha) = -D^{-1}\left(\alpha L + \frac{1}{\alpha} U\right)$$ are independent of α.

$$C(1) = -D^{-1}(L+U) = I - D^{-1}A = P_J.$$

Thm Let, A is consistently ordered and $P_J = I - D^{-1}A$ has real eigenvalues and $\rho(P_J) < 1$. Then

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(P_J)^2}} \quad \text{and}$$

$$\rho(P_{SOR}) = \omega_{opt} - 1 = \frac{\rho(P_{SOR})^2}{\left(1+\sqrt{1-\rho(P_J)^2}\right)^2}$$

Exm : Tri-diagonal spd matrices.

— $\omega_{opt} \in (1,2)$.

Theorem GS method converges if A is strictly diagonally dominant.

Proof : GS :

$$x^{(k+1)} = -(L+D)^{-1}U x^{(k)} + (L+D)^{-1}b$$

This scheme will be convergent if

$$\rho\left(-(L+D)^{-1}U\right) < 1$$

$$P_{GS} = -(L+D)^{-1}\left(A-(L+D)\right)$$

$$= I - (L+D)^{-1}A.$$

Let, λ be an eigen value and x is ~~the~~ an eigen-vector of P_{GS} with $\|x\|_\infty = 1$.

Let, $|x_m| = 1$ & $|x_i| \leq 1 \ \forall \ i \neq m$.

$$(I-(D+L)^{-1}A)\,x = \lambda x$$

$$\Rightarrow (D+L)x - Ax = \lambda(D+L)x$$

$$\Rightarrow -\sum_{j>m} a_{mj}x_j = \lambda \sum_{j=1}^{m} a_{mj}x_j \quad \text{(mth equation)}$$

$$\Rightarrow \quad \lambda a_{mm} x_m = -\sum_{j>m} a_{mj} x_j - \lambda \sum_{j<m} a_{mj} x_j$$

$$\Rightarrow \quad |\lambda|\,|a_{mm}| \le \sum_{j>m} |a_{mj}|\,|x_j| + |\lambda| \sum_{j<m} |a_{mj}|\,|x_j|$$

$$\le \sum_{j>m} |a_{mj}| + |\lambda| \sum_{j<m} |a_{mj}|$$

$$\therefore \quad |\lambda| \le \frac{\sum_{j>m} |a_{mj}|}{|a_{mm}| - \sum_{j<m} |a_{mj}|} < 1$$ if A is strictly diag. dom.

So, $\underline{\rho(P_{GS}) < 1}$.