Defn :- Let, V be a vector space over $\mathbb{R}$. A mapping $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$ is called an inner product if it is

a) symmetric : $\langle x, y \rangle = \langle y, x \rangle \quad \forall\, x, y \in V$

b) linear : $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

$\alpha, \beta \in \mathbb{R}$,
$x, y, z \in V$

c) definite: $\langle x, x \rangle > 0 \,;\quad x \in V - \{0\}$.

The canonical norm is defined as:

$$\|x\| := \sqrt{\langle x, x \rangle}\,.$$

Lemma (Gram - Schmidt)

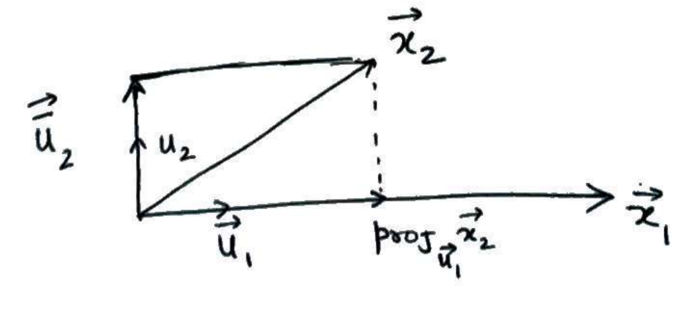
Let, $x_1, x_2, \ldots, x_n$ be L.I vectors of a vector space V. Define $u_1 = \dfrac{x_1}{\|x_1\|}$ and then for $j = 1, 2, \ldots, n-1$

$$\bar{u}_{j+1} = x_{j+1} - \sum_{i=1}^{j} \underbrace{\langle x_{j+1}, u_i \rangle u_i}_{\text{Proj}_{u_i} x_{j+1}}$$

$$u_{j+1} = \frac{\bar{u}_{j+1}}{\|\bar{u}_{j+1}\|}.$$

Then the set $\{u_1, u_2, \ldots u_k\}$ forms an orthonormal basis of span $\{x_1, x_2, \ldots, x_k\}$ for $k = 1, 2, \ldots, n$.

Graphical Explanation :-



$$\bar{u}_2 = \vec{x}_2 - \text{Proj}_{u_1} x_2$$
$$= x_2 - \langle x_2, u_1 \rangle u_1$$

EXM. $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}$. Apply Gram-Schmidt to $\{x_1, x_2, x_3\}$.

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\bar{u}_2 = x_2 - \langle x_2, u_1 \rangle u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$-1 \quad -2 \quad -1 \quad \overline{-4}$$

So, $$u_2 = \frac{1}{\|\bar{u}_2\|}\bar{u}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\bar{u}_3 = \cancel{u_3} \; x_3 - \langle x_3, u_2 \rangle u_2 - \langle x_3, u_1 \rangle u_1$$

$$= \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \end{pmatrix} + \cancel{2} \cdot \frac{1}{\cancel{2}}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - 1 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3 = \frac{\bar{u}_3}{\|\bar{u}_3\|} = \underline{\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}$$

Properties :-

i) $\bar{u}_{J+1} \perp u_i$ for $1 \le i \le J$ as

$$\langle \bar{u}_{J+1}, u_i \rangle = \langle x_{J+1}, u_i \rangle - \langle x_{J+1}, u_i \rangle = 0.$$

ii) $\|u_{J+1}\| = 1 \quad \forall J$.

QR Decomposition :-

Full QR decomposition of $A_{m\times n}$, $m \ge n$ means

$A = QR$ where Q is an $m\times m$ orthogonal matrix and R is an $m \times n$ upper triangular matrix.

$$A = QR = Q\begin{pmatrix} R_1 \\ 0 \end{pmatrix} = (Q_1 \;\; Q_2)\begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

$$= \underset{m\times n}{Q_1}\,\underset{n\times n}{R_1}$$

Alternative approach: (Reduced QR factorization)

Theorem 1:- Let, A be $m \times n$ with $m \geq n$. Suppose that A has full column rank. Then $\exists$ a unique $m \times n$ ~~orthogonal matrix~~ Q ($Q^TQ = I_n$) (A matrix Q having orthonormal columns) and a unique $n \times n$ upper triangular matrix R with positive diagonals $r_{ii} > 0$ s.t. $A = QR$.

Proof: This ~~statement~~ Theorem is a restatement of Gram-Schmidt orthogonalization process.

Let, $A = [a_1, a_2 \ldots a_n]$. We apply Gram-Schmidt from left to right to get a sequence of orthonormal vectors $\{q_1, q_2 \ldots, q_n\}$ spanning the same space.

$$\text{Let, } Q = [q_1, q_2, \ldots, q_n]$$

If we express $a_i = \sum_{j=1}^{i} r_{ji} q_j$, then by Gram-Schmidt process,

$$r_{ji} = \langle q_j, a_i \rangle \quad , \quad j \leq i, \quad 1 \leq i \leq n$$

$$\text{So, } R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ & r_{22} & r_{23} & & \vdots \\ & & r_{33} & & \vdots \\ & & & \ddots & \vdots \\ & & & & r_{nn} \end{pmatrix}_{n \times n}$$

$$\text{Since, } q_i = \frac{a_i - \sum_{k=1}^{i-1} \langle a_i, q_k \rangle q_k}{\| a_i - \sum_{k=1}^{i-1} \langle a_i, q_k \rangle q_k \|} \quad 1 \leq i \leq n$$

we have $a_i = \sum_{k=1}^{i} r_{ki} q_k$, with $r_{ki} = \langle q_k, a_i \rangle$

& $r_{ii} = \| a_i - \sum_{k=1}^{i-1} r_{ki} q_k \| > 0$.

This gives $A = QR$.

Solving $Ax = b$

$$QRx = b$$
$$\Rightarrow (Q^TQ)Rx = Q^Tb$$
$$\Rightarrow \underline{Rx = Q^Tb} \longrightarrow \text{Back-substitution.}$$

Let $A = Q_1R_1 = Q_2R_2$

$\Rightarrow Q_2^TQ_1 = R_2R_1^{-1}$ is an upper triangular matrix with +ve diagonal entries.

Let, $T = R_2R_1^{-1}$

$$TT^T = (Q_2^TQ_1)(Q_2^TQ_1)^T = Q_2^TQ_1Q_1^TQ_2 = I. \quad \text{(Wrong)}$$

So, T is a cholesky factorization of I. Since it is unique, $T = I$. i.e. $R_2 = R_1$ & $Q_1 = Q_2$.

Lemma: Let, $Q \in \mathbb{R}^{n\times n}$ be orthogonal & $A \in \mathbb{R}^{n\times n}$, then $\|Qx\|_2 = \|x\|_2 \;\forall x \in \mathbb{R}^n$ and $\|QA\|_2 = \|A\|_2$.

Proof:
$$\|Qx\|_2^2 = \langle Qx, Qx\rangle = (Qx)^T(Qx)$$
$$= x^Tx = \langle x, x\rangle = \|x\|_2^2$$

i.e. $\|Qx\|_2 = \|x\|_2$

$$\|QA\|_2 = \sup_{\|x\|_2=1} \|QAx\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 = \|A\|_2$$

Note: Let, A be an invertible matrix & $A = QR$.

Then $\|QR\|_2 = \|R\|_2 = \|A\|_2$

$$K_2(A) = \|A\|_2\underline{\|A^{-1}\|_2} = \|R\|_2\underline{\|R^{-1}\|_2} = K_2(R).$$

$A^{-1} = R^{-1}Q^T$.

$$\|A^{-1}\|_2 = \|R^{-1}Q^T\|_2 = \sup_{\|x\|_2=1} \|R^{-1}Q^T x\|_2$$

$$= \sup_{\|Q^T x\|_2=1} \|R^{-1}(Q^T x)\|_2$$

$$= \sup_{\|y\|_2=1} \|R^{-1}y\|_2 = \|R^{-1}\|_2 .$$

Uniqueness in Theorem 1 :-

Let, $A = Q_1R_1 = Q_2R_2$ $Q_1, Q_2 \in \mathbb{R}^{m\times n}$

$R_1, R_2 \in \mathbb{R}^{n\times n}$.

& $m \geq n$

So, $Q_2^T Q_1 R_1 = R_2$ ..(i)

& $Q_1^T Q_2 R_2 = R_1$.(ii)

Use (ii) in (i) to get:

$$(Q_2^T Q_1 Q_1^T Q_2) R_2 = R_2 \quad , \text{ Let, } T = Q_2^T Q_1 Q_1^T Q_2$$

$$\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ \vdots & & & \\ t_{i1} & t_{i2} & \cdots & t_{in} \\ t_{n1} & \cdots & & t_{nn} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

The ith row of R_2 can be written as a linear combination of all the rows of R_2 with co-eff from ith row of T. But R_2, being an upper triangular matrix with +ve diagonals is invertible. So, $\{r_1, r_2, \ldots r_n\}$ are indep. rows. This is possible only when

$$t_{ii} = 1 \ \& \ t_{ij} = 0 \quad \forall i \neq j .$$

i.e. $T = I$.

$$(Q_2^T Q_1)(Q_2^T Q_1)^T = I .$$

Now $Q_2^T Q_1$ is a square matrix, so an orthogonal matrix.

Now, $Q_2^T Q_1 = R_2 R_1^{-1}$

i.e. $(R_2 R_1^{-1})(R_2 R_1^{-1})^T = I$, a cholesky factorization

So, $R_2 R_1^{-1} = I \Rightarrow R_1 = R_2$.

So, $\underline{Q_1 = Q_2}$.

<u>Theorem 2</u>. If $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, then $\exists$ ~~an~~ ~~orthonormal~~ unique orthogonal Q & unique upper triangular matrix with +ve diagonal R such that $A = QR$.

<u>Householder Transformation</u>:-

A Householder transformation (or Reflection) is a matrix P given by

$$P = I - 2uu^T \quad \text{where } \|u\|_2 = 1.$$

Check: i) $P^T = P$

ii) $PP^T = (I - 2uu^T)(I - 2uu^T)$

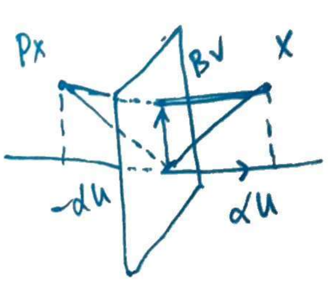
$$= I - 4uu^T + 4uu^Tuu^T$$

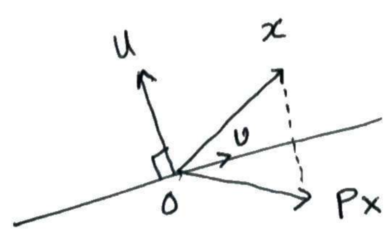
$$= I - 4uu^T + 4u\|u\|_2^2 u^T$$

$$= I.$$

So, P is symmetric, orthogonal matrix.

$\Rightarrow$ It is called a reflection as Px is reflection of x in the plane through 0 perpendicular to u





Let, $x = \alpha u + \beta v$ where $v \perp u$.

$Px = (I - 2uu^T)(\alpha u + \beta v)$

$= -\alpha u + \beta v$

So, the component of x in the direction of u has been reversed.

Theorem 3

For every vector x, $\exists\, u$ with $u = 0$ or $\|u\|_2 = 1$

such that $Px = \|x\|_2 e_1$

Proof: If $x = 0$ or $x = \|x\|_2 e_1$, then $u = 0$.

i.e. $P = I$.

Else, let, $u = \dfrac{x - \|x\|_2 e_1}{\|x - \|x\|_2 e_1\|_2}$ ... (1)

Then, $$1 = u^T u = \frac{(x - \|x\| e_1)^T (x - \|x\| e_1)}{\|x - \|x\| e_1\|_2^2}$$

$$= \frac{x^T x - 2\|x\|_2 x^T e_1 + \|x\|_2^2}{\|\cdot\|_2^2}$$

$$= \frac{2x^T (x - \|x\|_2 e_1)}{\|\cdot\|_2^2}$$

$$= \frac{2x^T u}{\|x - \|x\|_2 e_1\|_2}$$

So that, $2x^T u = \|x - \|x\|_2 e_1\|_2$

So, $$Px = (I - 2uu^T)x$$

$$= x - 2(u^T x) u$$

$$= x - \|x - \|x\|_2 e_1\|_2 u$$

$$Px = \|x\|_2 e_1 \quad \text{from (1)}.$$

Note: (i) $Px = \begin{pmatrix} \|x\|_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

(ii) If $u = \dfrac{x + \|x\| e_1}{\|\cdot\|_2}$, then $Px = -\|x\|_2 e_1$

Given two vectors $x, y \in \mathbb{R}^n$, $\|x\|_2 = \|y\|_2 = 1$, $\exists\, P = I - 2uu^T$ s.t $Px = y$.

Take $u = \dfrac{x-y}{\|x-y\|_2}$ & $P = I - 2uu^T$.

Check: $\underline{Px = y}$.

Let, $v = x - y$ i.e. $u = \dfrac{v}{\|v\|_2}$ $\quad P = I - 2\dfrac{vv^T}{v^Tv}$

$$v^Tv = x^Tx + y^Ty - 2x^Ty = 2(1 - x^Ty)$$

$$v^Tx = x^Tx - y^Tx = \frac{1}{2}v^Tv$$

So, $Px = \left(I - 2\dfrac{vv^T}{v^Tv}\right)x$

$$= x - 2\frac{(v^Tx)}{v^Tv}v$$

$$= x - v = y.$$

Householder Algorithm :-

$$A = \begin{pmatrix} \times & \times \\ \times & \times \\ \times & \times \end{pmatrix}_{3\times 2}$$

S1 $\quad P_1A = \begin{pmatrix} \times & \times \\ 0 & \times \\ 0 & \times \end{pmatrix} = A_1$

$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & \\ 0 & P_2' \end{pmatrix}$ so that

S2 $\quad P_2A_1 = \begin{pmatrix} \times & \times \\ 0 & \times \\ 0 & 0 \end{pmatrix} = A_2$

So, $P_2P_1A = A_2 = R$

$\Rightarrow A = P_1^TP_2^TR = P_1P_2R$ as P_1, P_2 are symmetric.

$= QR.$

Th Exa

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix} = \left[a_1 \; a_2 \; a_3 \right]$$

Let $v_1 = a_1 - \|a_1\| e_1$ so that $u_1 = \dfrac{v_1}{\|v_1\|_2}$

and $P_1 a_1 = (I_3 - 2u_1 u_1^T) a_1 = \begin{pmatrix} \|a_1\|_2 \\ 0 \\ 0 \end{pmatrix}$

$$v_1 = \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix} \quad \text{so that} \quad P_1 = \begin{pmatrix} 6/7 & 3/7 & -2/7 \\ 3/7 & -2/7 & 6/7 \\ -2/7 & 6/7 & 3/7 \end{pmatrix}$$

$$A_1 = P_1 A = \begin{pmatrix} 14 & 21 & -14 \\ 0 & -49 & -14 \\ 0 & 168 & -77 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -49 \\ 168 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & P_2' \end{pmatrix} \quad \text{where} \quad P_2' = I_2 - 2u_2u_2^T$$

with $u_2 = \dfrac{v_2}{\|v_2\|_2}$, $v_2 = a_2 - \|a_2\|$

$$= \begin{pmatrix} -224 \\ 168 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -7/25 & \frac{24}{25} \\ 0 & \frac{24}{25} & 7/25 \end{pmatrix}$$

$$A_2 = P_2 A_1 = \begin{pmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & -35 \end{pmatrix}$$

So $$A = (P_1 P_2) A_2 = \begin{pmatrix} 6/7 & -\frac{69}{175} & \frac{58}{175} \\ 3/7 & \frac{158}{175} & -\frac{6}{175} \\ -2/7 & \frac{6}{35} & \frac{165}{175} \end{pmatrix} \begin{pmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & -35 \end{pmatrix}$$

$$= QR.$$

Q is orthogonal & R is upper triangular.

H.W Write QR for $A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 6 & 3 \\ 0 & 3 & 7 \end{pmatrix}$

Theorem For every $A \in \mathbb{R}^{m\times n}$, $m \geq n$, $\exists$ an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and a upper triangular matrix $R \in \mathbb{R}^{m\times n}$ such that $A = QR$.

Linear Least Square Problem :-