Linear Least Square Problem :-

$$\underset{m\times n}{A}\, x = b$$

$m<n$: under-determined system : infinite sol^n

$m=n$: A invertible : unique sol^n $x = A^{-1}b$.

$m>n$: overdetermined system : no exact sol^n

So, we look for 'generalized solutions' such that Ax is ~~closest~~ 'closest possible' to b.

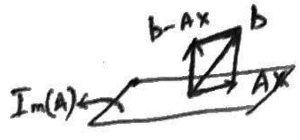
Problem: To minimize $\|b - Ax\|_2$ where $A \in M_{m,n}(\mathbb{R})$

$x \in \mathbb{R}^n$

$b \in \mathbb{R}^m$.

i.e. $$\|b - Ax\|_2 = \min_{y \in \mathbb{R}^m} \|b - Ay\|_2 .$$

~~the~~ Claim: $(b - Ax)$ is $\perp$ $\mathrm{Im}(A) = \{Ay : y \in \mathbb{R}^n\}$



i.e. $\langle b - Ax, Az \rangle = 0$

Theorem:

A vector $x \in \mathbb{R}^n$ is a solution of the least squares problem iff it satisfies the normal equation:

$$A^T A x = A^T b.$$

Proof:

Let, $x \in \mathbb{R}^n$ solves

$$\min_{y \in \mathbb{R}^n} \|b - Ay\|_2 \quad \text{i.e.} \quad \ldots (1)$$

$$\|b - Ax\|_2^2 \leq \|b - Ay\|_2^2 \quad \forall\, y \in \mathbb{R}^n$$

For any $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$, set $y = x + tz$

Then,

$$\|b - Ax\|_2^2 \leq \langle b - Ay, b - Ay \rangle$$

$$= \langle b - Ax, b - Ax \rangle + t^2 \langle Az, Az \rangle + 2t \langle b - Ax, Az \rangle$$

i.e. $$t^2 \|Az\|_2^2 + 2t \langle b - Ax, Az \rangle \geq 0$$

$$\Rightarrow \quad |t| \, \|Az\|_2^2 + 2\,\mathrm{sgn}(t) \langle b - Ax, Az \rangle \geq 0$$

So, as $t \to 0+$ & $t \to 0-$ we get $\forall\, t \in \mathbb{R}$, $z \in \mathbb{R}^n$.

$$\langle b - Ax, Az \rangle = 0.$$

i.e. $$z^T A^T (b - Ax) = 0 \quad \forall\, z \in \mathbb{R}^n$$

$$\therefore \quad A^T A x = A^T b$$

Conversely, let x is a sol$^{\underline{n}}$ of the normal eq$^{\underline{n}}$.

Then, $\langle b - Ax, Az \rangle = 0 \quad \forall\, z \in \mathbb{R}^n$.

Thus $\|b - Ax\|_2^2 \leq \|b - Ay\|_2^2$, $\forall\, y = x + tz \in \mathbb{R}^n$.

Therefore, x is a sol$^{\underline{n}}$ of (1).

Note:

$$\text{rank}(A) = \text{rank}(A^TA)$$

As $x \in N(A) \Rightarrow Ax = 0$

$\Rightarrow A^TAx = 0$

$\Rightarrow x \in N(A^TA)$ So, $N(A) \subseteq N(A^TA)$

& $x \in N(A^TA) \Rightarrow A^TAx = 0$

$\Rightarrow x^TA^TAx = 0$

$\Rightarrow \langle Ax, Ax \rangle = 0$

$\Rightarrow Ax = 0$

$\Rightarrow x \in N(A)$ So, $N(A^TA) \subseteq N(A)$

Since, $\text{rank}(A) + \dim(N(A)) = n$, it concludes.

$\Rightarrow$ Now if $\text{rank}(A) = n$, then A^TA is non-singular

So, $A^TAx = A^Tb \Rightarrow x = (A^TA)^{-1}A^Tb$

$= A^+b$.

$\Rightarrow$ $A^+ := (A^TA)^{-1}A^T$ is called the pseudo-inverse of A.

Theorem. The solution of the least squares problem (1) with $\text{rank}(A) = n$ is the unique solution of the normal equation: $A^TAx = A^Tb$.

##

$$\begin{pmatrix} 1 & 2 \\ 1 & 3/2 \\ 1 & 4 \end{pmatrix}_{3\times 2} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$ does not have a soln. as there is no line $y = c + dx$ that passes through $(2,1)$, $(3/2, 2)$, $(4,1)$.

Thm.

$Im(A) = N(A^T)^\perp$ Orthogonal Complement.

$$W^\perp := \{ v : \langle v, w \rangle = 0 \ \forall w \in W \}$$

Proof:

Let, $y \in Im(A)$ $A_{m \times n}$

$\exists x \in \mathbb{R}^n$ s.t. $y = Ax$.

For any $v \in N(A^T)$, $A^T v = 0$.

So, $y^T v = x^T A^T v = 0$

$\Rightarrow y \in N(A^T)^\perp$

So, $Im(A) \subseteq N(A^T)^\perp$.

To prove $N(A^T)^\perp \subseteq Im(A)$

i.e. to prove $Im(A)^\perp \subseteq N(A^T)$.

Let, $x \in Im(A)^\perp$.

i.e. for any $y \in Im(A)$, $x^T y = 0$.

Now, $Au \in Im(A)$ for any $u \in \mathbb{R}^n$

So, $x^T A u = 0$

i.e. $u^T A^T x = 0 \quad \forall u \in \mathbb{R}^n$

i.e. $A^T x = 0 \Rightarrow x \in N(A^T)$

So, $Im(A)^\perp \subseteq N(A^T)$

So, $Im(A) = N(A^T)^\perp$.

Theorem 1. $Ax = b$ has a solution iff $A^T v = 0 \Rightarrow b^T v = 0$.

Proof: Let $Ax = b$ has a solution $x^* \in \mathbb{R}^n$.

So, $Ax^* = b$

$\Rightarrow x^{*T} A^T = b^T$.

$\Rightarrow (x^*)^T A^T v = b^T v$ for any $v \in \mathbb{R}^m$.

Now, if $A^T v = 0$ for some v

then $b^T v = 0$.

Suppose $A^T v = 0 \Rightarrow b^T v = 0$

i.e. if $v \in N(A^T) \Rightarrow b^T v = 0$

Therefore $b \in N(A^T)^{\perp} = Im(A)$.

So, $\exists\, x \in \mathbb{R}^n$ s.t. $\underline{b = Ax}$.

Theorem 2. The normal eqn has at least one solution.

Proof: By Theorem 1, $My = c$ has a soln iff

$M^T v = 0 \Rightarrow c^T v = 0$.

Let, $M = A^T A$, $c = A^T b$.

$M^T v = 0 \Rightarrow A^T A v = 0 \Rightarrow v \in N(A^T A) = N(A)$

So, $Av = 0$.

Now, $c^T v = b^T A v = 0$.

Therefore, $A^T A x = A^T b$ has a soln.

Theorem 3 If $\operatorname{rank}(A) < n$, then the normal eqn has infinite solutions and any two solutions x_1 & x_2 satisfy $A(x_1 - x_2) = 0$.

Proof :- If x_1 & x_2 solve $A^T A x = A^T b$. then

$$A^T A (x_1 - x_2) = 0$$

$$\Rightarrow \quad x_1 - x_2 \in N(A^T A) = N(A)$$

$$\Rightarrow \quad A(x_1 - x_2) = 0$$

Since $\operatorname{ran}(A) < n$, ~~ker~~ $\dim(N(A)) > 1$.

So, $\exists \; v \neq 0$ s.t $Av = 0$

If $\mathbf{x}$ is any soln of the normal eqn then $\bar{x}_\lambda = x + \lambda v$ is another soln. So, we can have infinite soln.

We discuss two ways to compute the solution of the least square problem:

$$\min_{x \in \mathbb{R}^n} \| b - Ax \|_2$$

A. QR Method:

Let, $A = \underset{m \times m}{Q} \underset{m \times n}{R}$ (Householder)

Q orthogonal; $Q^T Q = Q Q^T = I_m$

$$\begin{aligned}
\| Ax - b \|_2^2 &= \| QRx - b \|_2^2 \\
&= \| Rx - Q^T b \|_2^2 \\
&= \left\| \begin{pmatrix} R_1 \\ 0 \end{pmatrix} x - \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right\|_2^2 \\
&= \| R_1 x - c_1 \|_2^2 + \| c_2 \|_2^2
\end{aligned}$$

where $c = Q^T b \in \mathbb{R}^m$

$R_1 \in \mathbb{R}^{n\times n}$ be the upper part of the upper triangular matrix. R.

Since, $\|c_2\|_2^2$ is constant, $\|Ax-b\|_2^2$ can be minimized if $R_1 x = c_1$ is solved.

This can be solved by Back-substitution.

B. Singular Value Decomposition (SVD)

Lemma: A^TA has real non-negative eigenvalues.

Proof: Let, λ be any eigen-value of A^TA with $x \neq 0$ be a corresponding eigen-vector.

Then, $$A^TAx = \lambda x$$

$$\Rightarrow x^TA^TAx = \lambda x^Tx.$$

$$\Rightarrow \|Ax\|_2^2 = \lambda \|x\|_2^2$$

$$\therefore \lambda = \frac{\|Ax\|_2^2}{\|x\|_2^2} \geq 0.$$

Definition: The singular values of $A \in M_{m\times n}(\mathbb{R})$ are the non-negative square roots of the n eigenvalues of A^TA.

Lemma: Let, $A \in M_{m\times n}(\mathbb{R})$ & $B \in M_{n\times m}(\mathbb{R})$. The non-zero eigen-values of AB & BA are the same.

Proof: Let, λ be an eigen-value of AB & u is a corresponding eigen-vector.

$$ABu = \lambda u.$$

If $u \in N(B) \Rightarrow Bu = 0 \Rightarrow \lambda u = 0 \Rightarrow \lambda = 0$.

So, if $\lambda \neq 0$, then $u \notin N(B)$. So, $Bu \neq 0$.

$$ABu = \lambda u$$

$$\Rightarrow BA(Bu) = \lambda(Bu)$$

$\Rightarrow$ λ is also an eigen-value of BA.

So, the non-zero eigen-values of A^TA & AA^T are the same.

<u>Alternative Defn:</u> The non-negative square roots of the m eigen-values of AA^T are the singular values of A.

<u>Theorem</u> (SVD) Let, $A \in M_{m\times n}(\mathbb{R})$ be a matrix having r +ve singular values σ_i. $\exists$ two orthogonal matrices $U \in M_{m\times m}(\mathbb{R})$ & $V \in M_{n\times n}(\mathbb{R})$ such that $A = U\Sigma V^T$ where

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r, 0 \ldots 0)$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$.

<u>Proof:</u> Let, v_i be the eigen-vectors of A^TA corresponding to the eigen-values σ_i^2.

$$A^TAv_i = \sigma_i^2 v_i$$

and $V = [v_1 \; v_2 \ldots v_n]_{n\times n}$. As the set $\{v_1, v_2, \ldots v_n\}$ is orthonormal, $V^TV = VV^T = I_n$.

We have: $$A^TAV = [A^TAv_1 \; \ldots \; A^TAv_n]$$
$$= [\sigma_1^2 v_1 \; \sigma_2^2 v_2 \ldots \sigma_n^2 v_n]$$
$$= V\,\text{diag}(\sigma_1^2, \sigma_2^2 \ldots \sigma_n^2)$$

So, $$V^TA^TAV = \Sigma^T\Sigma$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdot & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & \cdot & \cdot & \sigma_n \\ 0 & \cdot & & 0 \\ \cdot & & & \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{m \times n}$

We arrange $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$, of which only the first r are non-zero.

Also, $\underset{\langle Av_i, Av_j \rangle}{=} v_i^T A^T A v_j = \sigma_j^2 v_i^T v_j = \begin{cases} \sigma_i^2 & \text{if } i = j,\ i \leq r \\ 0 & \text{if } i \neq j,\ i > r \end{cases}$

In particular $Av_i = 0$ for $r < i \leq n$.

For $1 \leq i \leq r$, $\sigma_i \neq 0$, so we define

$$u_i = \frac{Av_i}{\sigma_i} \in \mathbb{R}^m$$

Check that $\langle u_i, u_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle Av_i, Av_j \rangle = \begin{cases} 1, & \text{if } i = j,\ i \leq r \\ 0 & \text{if } i \neq j \\ & i > r \end{cases}$

Extend the set $\{u_1, u_2, \ldots, u_r\}$ to set a full set of orthognormal set in $\mathbb{R}^m$.

Define $U = [u_1\ u_2\ \ldots\ u_m]_{m \times m}$ so that

$U^T U = U U^T = I_m$.

Now $U\Sigma = [u_1\ u_2\ \ldots\ u_m]_{m \times m} \cancel{\text{diag}(\sigma_1\ \sigma_2} \ \Sigma_{m \times n}$

$= [\sigma_1 u_1\ \ \sigma_2 u_2\ \cdot\ \sigma_n u_n]_{m \times n}$

$= [Av_1\ Av_2\ \ldots\ Av_r\ 0 \cdots 0]$

$= AV \implies A = U \Sigma V^T$

Note -- 1. $A = U\Sigma V^T$

The eigen-vectors of A^TA are in the column of V and that of AA^T are in " " U.

$AA^T = U\Sigma\Sigma^T U^T$ $\Rightarrow$ U must be the eigen-vector matrix of AA^T.

2. Polar Decomposition :

Every real square matrix can be factored as $A = QS$, Q orthogonal & S symmetric positive semi-definite. If A is investible, S is spd.

$$A = U\Sigma V^T$$
$$= UV^T(V\Sigma V^T)$$
$$= QS$$

S is symm. positive semidefinite as is " Σ

$Q = UV^T$ is orthogonal as

$$Q^TQ = VU^TUV^T = I$$

3. $A^+ = (A^TA)^{-1}A^T$, pesudo inverse if rank $(A^TA) = n$.

Then, $A = U\Sigma V^T$ with $\Sigma =$ ~~diag(~~ $\begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ & & \cdots & \sigma_n \\ 0 & & \text{---} & 0 \\ \vdots & & & \\ 0 & & \text{---} & 0 \end{pmatrix}_{m\times n}$

Now $A^TA = V\Sigma^T\Sigma V^T$ so that

$$A^+ = (V^T)^{-1}(\Sigma^T\Sigma)^{-1}V^TV\Sigma^TU^T$$
$$= V(\Sigma^T\Sigma)^{-1}\Sigma^TU^T$$
$$= V\Sigma^+U^T$$

where $\Sigma^+ = \begin{pmatrix} 1/\sigma_1 & & & 0 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & & & \vdots & & \vdots \\ & & 1/\sigma_r & & & & \\ 0 & & 0 & 1/\sigma_n & 0 & \cdots & 0 \end{pmatrix}_{n \times m}$

$A^+ = V \Sigma^+ U^T$ holds even if $\text{rank}(A^T A) < n$.

Theorem Let, $A = U \Sigma V^T$. Then

$$\|A\|_2 = \sigma_1 \quad \& \quad \|A\|_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

Proof: U & V are orthogonal $\Rightarrow$

$$\|A\|_2 = \|U \Sigma V^T\|_2 = \|\Sigma\|_2$$

Now,
$$\|\Sigma\|_2^2 = \max_{\|x\|_2 = 1} \|\Sigma x\|_2^2$$
$$= \max_{\|x\|_2 = 1} \left(\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \cdots + \sigma_n^2 x_n^2\right)$$
$$\leq \sigma_1^2 \left(x_1^2 + x_2^2 + \cdots + x_n^2\right) = \sigma_1^2$$

and the max is attained for $x = e_1$. So, $\|A\|_2 = \sigma_1$.

$$\|A\|_F = \sqrt{\sum_{ij} a_{ij}^2} = \sqrt{tr(A^T A)} = \sqrt{tr(V \Sigma^T \Sigma V^T)}$$
$$= \sqrt{tr(\Sigma^T \Sigma)} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}.$$

(Trace of a matrix is the sum of eigen-values)

Note:

(I) $A^T A v_i = \sigma_i^2 v_i$ $\quad i = 1(1)r$, $\quad A^T A v_i = 0$ $\quad i = r+1, \ldots, n$

(II) $A A^T u_i = \sigma_i^2 u_i$ $\quad i = 1(1)r$, $\quad A A^T u_i = 0$ $\quad i = r+1, \ldots, m$

(III) $A v_i = \sigma_i u_i$ $\quad i = 1(1)r$ $\quad A v_i = 0$ $\quad i = r+1, \ldots, n$

Moore Penrose Inverse :-

$Y = A^{+}_{n\times m}$ is the only solution of

a) $(AY)A = A$ b) $(YA)Y = Y$

c) $(AY)^T = AY$ d) $(YA)^T = YA$

for a matrix $A \in \mathbb{R}^{m\times n}$.

Verify $A = U\Sigma V^T$ & $A^{+} = V\Sigma^{+}U^T$ satisfy a), b), c) & d).

Least Squares Problem :-

$$\underset{m\times n}{A} = U\Sigma V^T.$$

$$\sigma_1 \geq \sigma_2 \geq \qquad \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_n.$$

Let, $y = V^T x$ & $c = U^T b$.

Then.
$$\|b - Ax\|_2^2 = \|UU^T b - U\Sigma V^T x\|_2^2$$
$$= \|U^T b - \Sigma V^T x\|_2^2$$
$$= \|c - \Sigma y\|_2^2$$
$$= \sum_{J=1}^{r} (c_J - \sigma_J y_J)^2 + \sum_{r+1}^{m} c_J^2$$
$$= \sum_{J=1}^{r} |u_J^T b - \sigma_J v_J^T x|^2 + \sum_{r+1}^{m} |u_J^T b|^2$$

So, the solution is: $y_J = v_J^T x = \frac{1}{\sigma_J} u_J^T b \quad 1 \leq J \leq r$.

The remaining y_J are free to choose.

Theorem. The general solution of the least square problem is given by:

$$x = \sum_{j=1}^{r} \frac{u_j^T b}{\sigma_j} v_j + \sum_{j=r+1}^{n} \alpha_j v_j$$

$$= x^+ + \sum_{r+1}^{n} \alpha_j v_j$$

where σ_j denotes the singular values of A, $v_j \in \mathbb{R}^n$ are the columns of V, $u_j \in \mathbb{R}^m$ are the columns of U from the SVD decomposition. The co-efficients α_j are free to choose. From all possible solutions, x^+ is the solution having minimal Euclidean norm.

Note:

$$A^+ b = V \Sigma^+ U^T b = V \Sigma^+ c = x^+$$

Result:

$A^+ \in \mathbb{R}^{n \times m}$ maps every vector $b \in \mathbb{R}^m$ to its norm-min least squares solution $x^+ \in \mathbb{R}^n$ i.e.

$$A^+ b = \cancel{\min} \left\{ x : \min\{\|x\|_2\} \ \& \ x \text{ solves } \min_z \|Az - b\|_2 \right\}$$

* Note: If $M_{n\times n}$ is symmetric, then

(i) all its eigen-values are real

(ii) eigen-vectors corresponding to distinct eigen-values are orthogonal.

i.e $\exists$ ~~as~~ a set $\{v_1, v_2, \ldots v_m\}$ of orthonormal eigen-vectors corresponding to distinct eigen-values $\lambda_1, \lambda_2, \ldots, \lambda_m$.

(iii) there are n orthonormal set of eigen-vectors even if there are repeated eigen-values.

Proof of (iii):-

Let, $\lambda_1, \lambda_2, \ldots \lambda_m$ are distinct eigen-values of M with corresponding eigen-vectors $v_1, v_2, \ldots, v_m$.

WLOG, let $\{v_1, v_2, \ldots, v_m\}$ is orthonormal set.

Set, $V = [v_1, v_2 \ldots v_m]_{n\times m}$, Then $V^TV = I_m$.

We have $AV = VD$ with

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_m)$$

So, $V^TAV = D$ & $V^TA = DV^T$.

Consider

$$\hat{A} = (I_n - VV^T)A(I_n - VV^T)$$

$$= A - VV^TA - AVV^T + VV^TAVV^T$$

$$= A - VDV^T - VDV^T + VDV^T$$

$$\underline{\hat{A}_{n\times n} = A - VDV^T}.$$

Note that $\hat{A}$ is symmetric, so has a real eigen-value λ and associated real eigen-vector x, so that

$$\hat{A}x = (A - VDV^T)x = \lambda x.$$

Pre-multiplying by V^T, we get

$$V^T A x - V^T V D V^T x = \lambda V^T x$$

$$\Rightarrow \quad \lambda V^T x = (V^T A - D V^T) x = 0.$$

Case I: $\lambda \neq 0$

Then $V^T x = 0$. $\Rightarrow$ ~~$V_{m+1} = \frac{x}{\|x\|}$~~

$$\Rightarrow \quad Ax = VDV^T x + \lambda x = \lambda x$$

$\Rightarrow$ λ is an eigen-value of A & $\hat{A}$ and x is an corresponding eigen-vector.

Also, $V^T x = 0 \Rightarrow x$ is orthogonal to each v_i

So, if $v_{m+1} = \frac{x}{\|x\|}$, then $\{v_1, v_2, \ldots, v_{m+1}\}$ constitutes a set of orthonormal eigen-vectors of A.

Case - II

$\lambda = 0$, i.e. the only eigen-vector of $\hat{A}$ is zero.

Then, by Rayleigh Quotient properties, or diagonalization of $\hat{A}$, $\hat{A} = 0$

i.e. $A = VDV^T$

Take any $x \neq 0$ such that $V^T x = 0$

This implies $Ax = VDV^T x = 0 = 0.x$

So, x is an eigen-vector of A corresponding to the eigen-value 0.

In this case also, $v_{m+1} = \frac{x}{\|x\|}$ yields $(m+1)$ orthonormal eigen-vectors of A. Repeat this process to get a complete list of n orthonormal eigen-vectors of A.

Image Processing :- Suppose a picture consists of 1000×1000 arrays of pixels. This can be thought of a 1000×1000 matrix A of numbers that represents colours.

Let, $A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$

a sum of rank-one matrices.

Suppose we take first 20 singular values. Then we send $20 \times 2000 = 40K$ numbers rather than 10^6 numbers.

This represents an image compression. $(25:1)$

i.e. 25Mb file size becomes 1Mb

Polar Decomposition :-

$$A = QS \qquad \text{with } Q = UV^T$$
$$S = V\Sigma V^T.$$

Also, $A^TA = S^TQ^TQS$

$= S^2.$

since A^TA is psd, $S = \sqrt{A^TA}$

$= V\Sigma V^T.$

and if $\text{rank}(A) = n$, then $\underline{Q = AS^{-1}}$

ExM.

$$A = \begin{pmatrix} 11 & -5 \\ -2 & 10 \end{pmatrix}, \quad A^TA = \begin{pmatrix} 125 & -75 \\ -75 & 125 \end{pmatrix}$$

so that $\lambda_i(A^TA) = 200, 50.$

So, $\sigma_1(A) = 10\sqrt{2}, \quad \sigma_2(A) = 5\sqrt{2}$

& $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So, $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix}$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So, $U = \begin{pmatrix} 4/5 & \frac{3}{5} \\ -3/5 & \frac{4}{5} \end{pmatrix}$

$$Q = UV^T = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix}$$

$$S = V\Sigma V^T = \frac{5}{\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

& $\underline{A = QS}$

<u>H.W.</u> Polar decomposition of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & -\frac{b}{\sqrt{*}} \\ \frac{b}{\sqrt{*}} & \frac{a}{\sqrt{*}} \end{pmatrix}\begin{pmatrix} \sqrt{*} & 0 \\ 0 & \sqrt{*} \end{pmatrix}$

A non-zero complex no has the form

$$z = re^{i\theta}$$

with $\underbrace{r \geq 0}_{S \text{ psd}}$ & $\underbrace{|e^{i\theta}| = 1}_{\text{unitary.}}$

<u>Hessenberg Form :-</u>

An $n \times n$ matrix A is called an upper Hessenberg matrix if $A_{ij} = 0 \quad \forall\, i > j+1$.

A matrix A is lower Hessenberg if A^T is upper Hessenberg.

$$A = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ 9 & 1 & 3 \end{pmatrix}$$

UH LH.

<u>Theorem</u> For any $n \times n$ matrix A, $\exists$ an orthogonal matrix P such that P^TAP is upper Hessenberg.

<u>Hint:</u> Use Householder transformation.

Exm

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix}$$

Let $a_1 = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

$$V_1 = a_1 - \|a_1\| e_1$$

$$= \begin{pmatrix} 6 - 2\sqrt{13} \\ -4 \end{pmatrix} \simeq \begin{pmatrix} -1.21 \\ -4 \end{pmatrix}$$

$$\& \; u_1 = \frac{V_1}{\|V_1\|_2} = \begin{pmatrix} -0.29 \\ -0.96 \end{pmatrix}$$

$$P_1 = I_2 - 2u_1u_1^T = \begin{pmatrix} 0.8318 & -0.5568 \\ -0.5568 & -0.84 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 1 & 0 \quad 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & P_1 \end{pmatrix}$$ so that

$$Q_1 A = \begin{pmatrix} 12 & -51 & 4 \\ 7.21 & 125.54 & -33.73 \\ 0 & -113.15 & 72.30 \end{pmatrix}$$

$$\& \; Q_1 A Q_1^T = \begin{pmatrix} 12 & -44.65 & 25.04 \\ 7.21 & 123.21 & -41.57 \\ 0 & -134.37 & 2.27 \end{pmatrix}$$

which is an upper Hessenberg form.

- If A is symmetric, then ~~it~~ this process reduces to a tri-diagonal matrix.

- The product of an upper Hessenberg matrix with tridiagonal matrix is upper Hessenberg.

- ~~QAQ^T is not applicable to make A upper triangular~~

  The same algorithm does not work to make A upper triangular. Check : $A = \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}$

Power Method

Numerical Method to compute the largest eigenvalue and corresponding eigen-vector. It is designed for matrices A with one dominant eigen-value λ_1 i.e.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|.$$

Assume that $\exists$ a basis $\{\omega_1, \omega_2, \dots, \omega_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A. Then any $x \in \mathbb{R}^n$ can be written as:

$$x = \sum_{j=1}^{n} \alpha_j \omega_j$$

So,

$$A^m x = \sum_{j=1}^{n} \alpha_j \lambda_j^m \omega_j = \lambda_1^m \left(\alpha_1 \omega_1 + \sum_{j=2}^{n} \left(\frac{\lambda_j}{\lambda_1}\right)^m \alpha_j \omega_j\right)$$

$$= \lambda_1^m \left(\alpha_1 \omega_1 + R_m\right)$$

where, $R_m \to 0$ as $m \to \infty$. If $\alpha_1 \neq 0$

then $\dfrac{A^m x}{\lambda_1^m} \to \alpha_1 \omega_1$ as $m \to \infty$

The later is an eigen-vector of A corresponding to λ_1

- λ_1 is not known, so $\dfrac{A^m x}{\lambda_1^m}$ cannot be formed.

Remedy:

$$\frac{\|A^{m+1} x\|_2}{\|A^m x\|_2} = |\lambda_1| \frac{\|\alpha_1 \omega_1 + R_{m+1}\|_2}{\|\alpha_1 \omega_1 + R_m\|_2} \to |\lambda_1| \quad \text{as} \quad m \to \infty$$

$\Rightarrow$

$A^m x \cong \lambda_1^m \alpha_1 x_1$ so that $A^m x$ approaches a multiple of the dominant eigen-vector of A.

Theorem If x is an eigen-vector of A, then its corresponding eigen-value is $\lambda = \dfrac{x^T A x}{x^T x}$.

The quotient $\dfrac{x^T A x}{x^T x}$ is called the Rayleigh quotient

Proof We have $Ax = \lambda x$

So, $\dfrac{x^T A x}{x^T x} = \dfrac{x^T(\lambda x)}{x^T x} = \lambda$.

Method 1 :

Start with $x_0 \neq 0$

1. Compute $A^m x_0$ with scaling to get eigen-vector x.
2. Use Rayleigh quotient to get corresponding λ.

Exm

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Ax_0 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 0.60 \\ 0.20 \\ 1 \end{pmatrix} = 5x_1$$

$$Ax_1 = \begin{pmatrix} 1 \\ 1 \\ 2.20 \end{pmatrix} = 2.20 \begin{pmatrix} 0.45 \\ 0.45 \\ 1 \end{pmatrix} = 2.20\, x_2$$

$$Ax_2 = 2.80 \begin{pmatrix} 0.48 \\ 0.55 \\ 1 \end{pmatrix} = 2.80\, x_3$$

$$Ax_3 = 3.13 \begin{pmatrix} 0.50 \\ 0.51 \\ 1 \end{pmatrix}$$

$$Ax_4 = 3.02 \begin{pmatrix} 0.50 \\ 0.50 \\ 1 \end{pmatrix}$$

So, $x = \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix}$ is the dominant eigen-vector and using

Rayleigh Quotient, $\lambda = \dfrac{x^T A x}{x^T x} = 3$ is the dominant eigen-value

Theorem 1. If A is an $n \times n$ diagonalizable matrix with a dominant eigen-value, then $\exists$ a non-zero vector x_0 s.t. the sequence $Ax_0, A^2x_0, A^3x_0 \ldots$ approaches a multiple of the dominant eigen-vector of A.

Method 2

Choose $x_0 \neq 0$ such that $\|x_0\| = 1$

Until $\delta_k = (x_k - x_{k-1}) < tol$

$$y_k = A x_{k-1}$$
$$x_k = \frac{y_k}{\|y_k\|}$$

Then, ~~as~~ $x_k \to x$ & ~~and~~ $\|y_k\| \to \lambda$ as $k \to \infty$

$$A\delta_k = Ax_k - Ax_{k-1}$$
$$= Ax_k - y_k$$
$$= Ax_k - \|y_k\| x_k$$

If $\delta_k \to 0$ as $k \to \infty$ $\quad Ax_k = \|y_k\| x_k$

Theorem 2. Let, A is diagonalizable with real eigen-values and the eigen-value with largest modulus is simple and positive

i.e. $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$

Then, the power method converges:

$$\lim_{k\to\infty} \|Ax_k\| = \lambda_1, \quad \lim_{k\to\infty} x_k = \pm v_1$$

Note: 1. If the largest eigen-value is -ve, we can apply PM to $-A$. If we blindly apply to A with -ve largest eigen-value, it is the sequence $(-1)^k x_k$ that converges, while $-\|y_k\|$ converges to the eigen-value.

2. It is possible to compute the smallest eigen-value of A by applying PM to A^{-1}. It is called the inverse power method.

<u>Method</u>: Choose $x_0 \neq 0$ with $\|x_0\| = 1$.

Until $\delta_k = x_k - x_{k-1} < tol$

Solve $Ay_k = x_{k-1}$

$$x_k = \frac{y_k}{\|y_k\|}$$

~~If $\delta_k \to 0$ as $k \to \infty$~~

Then, $x_k \to x$ and $\frac{1}{\|y_k\|} \to \lambda$ as $k \to \infty$.

$$A\delta_k = \frac{1}{\|y_k\|} x_{k-1} - Ax_{k-1} \to 0 \text{ as } k \to \infty$$

\# An eigen-vector is said to be normalized if the co-ordinate of the largest magnitude is unity.

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$y_k = Ax_{k-1}$$

$$x_k = \frac{y_k}{c_k}$$, c_k is the ~~mag~~ co-ordinate of y_k with largest magnitude.

Then, $\lim_{k \to \infty} x_k = v_1$ & $\lim_{k \to \infty} c_k = \lambda_1$

Lemma Suppose λ, v is an eigen pair of A. If $\alpha \neq \lambda$, then $\frac{1}{\lambda-\alpha}, v$ is an eigen-pair of $(A-\alpha I)^{-1}$

Theorem (Shifted Inverse Power Method)

Let, A has n distinct eigen-values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Choose α such that $\mu_1 = \frac{1}{\lambda_K - \alpha}$ is the dominant eigen-value of $(A-\alpha I)^{-1}$. Given $x_0 \neq 0$

$$y_K = (A-\alpha I)^{-1} x_{K-1}$$

$$x_K = \frac{1}{c_K} y_K. \quad \text{where,} \quad c_K = \max_{1 \le j \le n} |x_j^{(k)}|,$$

c_K & x_K converge to μ_1 & V_K of $(A-\alpha I)^{-1}$. Finally the corresponding eigen-value of the matrix A is given by

$$\lambda_K = \frac{1}{\mu_1} + \alpha.$$

Exm

$$A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}$$

$\lambda_1 = 4$
$\lambda_2 = 2$
$\lambda_3 = 1$

$\lambda_3 = 1$

$X_0 = (0,1,0)^T$.

Do

$$A y_K = x_{K-1}$$

$$x_K = \frac{1}{c_K} y_K$$

$$c_K = \max\{y_K(i)\}$$

Then $\frac{1}{c_K} \to 1$, $x_K \to \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$

$x_0 = (0,1,0)^T$

Do $(A - 2.1\, I) y_K = x_{K-1}$

$$x_K = \frac{1}{c_K} y_K$$

$$c_K = \max\{y_K(i)\}$$

Then $\frac{1}{c_K} + 2.1 \to 2$, $x_K \to \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$