I # If A is symmetric, then $\exists$ Q orthogonal such that

$$Q^T A Q = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

with $Q = [q_1\ q_2\ \ldots\ q_n]$ is formed with basis of of eigen-vectors q_i.

Now, $u = Q(Q^T u) = Q\omega = \sum_{i=1}^{n} q_i \omega_i$

$$\rho(u) = \frac{u^T A u}{u^T u}$$

$$= \frac{\omega^T Q^T A Q \omega}{\omega^T Q^T Q \omega}$$

$$= \frac{\omega^T D \omega}{\omega^T \omega} = \frac{\sum \lambda_i \omega_i^2}{\sum \omega_i^2}$$

This is the weighted average of the eigen-values of A.

If $\lambda_K = \max\{\lambda_i\}$ & $\lambda_m = \min\{\lambda_i\}$ then

$\omega = e_K$ & $\omega = e_m$ given

$$\underline{\max_{u \neq 0} \rho(u) = \lambda_K} \quad \& \quad \underline{\min_{u \neq 0} \rho(u) = \lambda_m}$$

i.e. $u = q_K$ & $u = q_m$ yields maxima & minima.

__Lemma.__ Let, A be a symmetric matrix with eigen-values $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le \lambda_n$ with corresponding eigen-vector $v_1, v_2, \ldots, v_n$. If S_K is a K-dimensional subspace then, $\exists$ $x \in S_K$ s.t.

$$\rho(x) \ge \lambda_K.$$

__Proof:__ Consider $S_K \cap \mathrm{span}\{v_K, v_{K+1}, \ldots v_n\}$.

This is non-empty. So, $\exists$ $x \in S_K$ with $x \in \mathrm{span}\{v_K, \ldots v_n\}$.

For this x, $r(x) = \dfrac{x^T A x}{x^T x}$

$$= \frac{d_1^T D_1 d_1}{d_1^T d_1} \qquad x = V_1 d_1$$

$$V_1 = [v_k, \dots v_n]$$

$$= \frac{\sum_{i=k}^{n} \alpha_i^2 \lambda_i}{\sum_{k}^{n} \alpha_i^2} \qquad D_1 = \operatorname{diag}(\lambda_k, \dots \lambda_n)$$

$$\geq \lambda_k.$$

Theorem (Min-Max Theorem)

Let, A be a symmetric matrix and suppose its eigen-values are $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then,

$$\min_{\dim S_k = k} \max_{x \in S_k} r(x) = \lambda_k.$$

Proof:- By previous lemma, $\min \max r(x) \geq \lambda_k$.

Now choose $S_k = \operatorname{span}\{v_1, v_2, \dots v_k\}$.

Then, $\max_{x \in S_k} r(x) \leq \lambda_k$

So, $\min_{\dim S_k = k} \max_{x \in S_k} r(x) = \lambda_k$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \begin{pmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x^T \begin{pmatrix} 1 & 5/2 & 5/2 \\ 5/2 & 2 & 3/2 \\ 5/2 & 3/2 & 2 \end{pmatrix} x.$$

$B^T = B$

$$A = \underline{\tfrac{1}{2}(A + A^T)} + \underline{\underline{\tfrac{1}{2}(A - A^T)}}$$

$r(x)$ depends only on symmetric part.

QR Algorithm :-

Let, $A \in \mathbb{R}^{n \times n}$.

Algorithm: Set $A_0 = A = Q_1 R_1$ (QR factorization)

Set $A_1 = R_1 Q_1 = Q_1^T A_0 Q_1$

$\quad \| $

$Q_2 R_2$

Set $A_2 = R_2 Q_2 = Q_2^T A_1 Q_2$

$\quad \|$

$Q_3 R_3$

In general
$$A_m = R_m Q_m = Q_m^T A_{m-1} Q_m$$

~~The~~ By construction, $A_m, A_{m-1}, \ldots A_1, A$ all are similar matrices, having same set of eigen-values.

Result: As m increases, A_m will ~~tend~~ tend towards a triangular or nearly triangular form.

Thus the eigen-values of A_m will be easy to calculate.

Exm. $A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$ $\qquad$ $\lambda_1 = 4, \lambda_2 = 9$

- $A_0 = A = \begin{pmatrix} 0.928 & 0.371 \\ 0.371 & 0.928 \end{pmatrix} \begin{pmatrix} -5.385 & 4.828 \\ 0 & 6.685 \end{pmatrix} = Q_1 R_1$

- $A_1 = R_1 Q_1 = \begin{pmatrix} 6.793 & -2.482 \\ -2.482 & 6.207 \end{pmatrix} = \begin{pmatrix} -0.939 & -0.343 \\ -0.343 & 0.939 \end{pmatrix} \begin{pmatrix} -7.233 & -4.462 \\ 0 & 4.977 \end{pmatrix}$

$= Q_2 R_2$

- $A_2 = R_2 Q_2 = \begin{pmatrix} 8.324 & -1.708 \\ -1.708 & 4.675 \end{pmatrix} = Q_3 R_3$

$\vdots$

$A_{12} = \begin{pmatrix} 8.999 & 0.00134 \\ 0.00134 & 4.00002 \end{pmatrix}$ $\qquad$ Approx eigen-values are on the diagonal.

Krylov Subspace Method :-

Stationary Iteration :

$$x^{(k)} = Px^{(k-1)} + c$$

P & c are independent of k.

Exm: Jacobi, GS, SOR.

Non-stationary Methods

CG, GMRES etc.

Krylov Subspace :- Given a matrix A, a vector b one can compute the sequence $\{b, Ab, A(Ab), A^2(Ab), \ldots\}$.

mth Krylov subspace is given by

$$K_m(A,b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}, \quad m \geq 1.$$

Using this subspace technique, following algorithms are developed:

	$Ax=b$	$Ax=\lambda x$
$A=A^T$	CG	Lanczos
$A \neq A^T$	GMRES, BCG	Arnoldi

By construction,

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \quad \text{nested subspaces.}$$

Motivation :

1. By Cayley-Hamilton theorem A^{-1} can be expressed as linear combination of powers of A. So, $A^{-1}b$, the solution lie in some K_r.

2. By Power Method $A^m x_0$ approximates the dominant eigen-vector.
This triggers finding eigen-vector in $K_r(A, x_0)$.

(1A) $A = \begin{pmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$p(t) = \det(A - tI) = 0$$

$$\Rightarrow -t^3 + 6t^2 + 8t - 41 = 0$$

C.H: $p(A) = 0$

$$\Rightarrow -A^3 + 6A^2 + 8A - 41I = 0$$

$$\Rightarrow A\left(\frac{1}{41}\left(-A^2 + 6A + 8I\right)\right) = I.$$

$$\therefore A^{-1} = -\frac{1}{41}A^2 + \frac{6}{41}A + \frac{8}{41}I.$$

$$\therefore x = A^{-1}b = \frac{8}{41}b + \frac{6}{41}Ab - \frac{1}{41}A^2b \in K_3(A, b)$$

Lemma: The sequence $\{K_m\}_{m \geq 1}$ is increasing. Moreover, for any vector $b \neq 0$, $\exists\ m_0 \in \{1, 2, \ldots, n\}$ such that

$$\dim(K_m) = \begin{cases} m, & 1 \leq m \leq m_0 \\ m_0, & m \geq m_0 \end{cases}$$

This integer m_0 is called the Krylov critical dimension.

Proof:- It is clear that $\dim(K_m) \leq m$ and $\dim(K_1) = 1$
Since $\dim(K_m) \leq n$ $\forall m$, $\exists\ m_0$, the greatest integer such that $\dim(K_m) = m$. $\forall m \leq m_0$
By definition of m_0, $\dim(K_{m_0+1}) < m_0+1$

But $K_{m_0} \subseteq K_{m_0+1}$ so we have $\dim(K_{m_0+1}) = m_0$

So, the vector $A^{m_0} b \in \text{span}\{b, Ab, \ldots, A^{m_0-1} b\} = K_{m_0}$

By induction argument, $A^m b \in K_{m_0} \quad \forall\ m \geq m_0$

Therefore, $K_m = K_{m_0} \quad \forall\ m \geq m_0$.

So, $$\dim(K_m) = \begin{cases} m, & 1 \leq m \leq m_0 \\ m_0, & m \geq m_0. \end{cases}$$

Two types of Iterative Methods:

$$Ax = b$$

(I) Stationary: $x_{k+1} = P x_k + C$, P & C are constants w.r.to k.

Exm: Jacobi, GS, SOR

(II) Non-stationary: $x_{k+1} = x_k + \alpha_k \underbrace{(b - A x_k)}_{\substack{r_k \\ \text{residue}}}$

α_k depends on iteration k.

Exm: CG, GMRES, BiCG [Developed using Krylov subspace.]

~~Lanczos Method~~:

(11) Arnoldi Method :-

It is a Krylov subspace iterative method that reduces A to upper-Hessenberg form.

* We apply Gram-Schmidt to generate an orthonormal basis $\{v_1, v_2, \ldots, v_m\}$ of $K_m(A, b)$.

We start with the similarity transformation:

$$A = QHQ^T$$

i.e. $$AQ = QH$$

If $m<n$, then the above eqn is written as:

$$A\underbrace{[v_1 \; v_2 \; \ldots \; v_n]}_{n\times n} = [v_1 \; v_2 \; \ldots \; v_n]\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1m} & \cdots & h_{1n} \\ h_{21} & h_{22} & & & & h_{2n} \\ & h_{32} & & & & \\ & & \ddots & h_{mm} & & \\ & & & h_{m+1,m} & & \\ & 0 & & & & \\ & & & & h_{n,n-1} & h_{nn} \end{bmatrix}$$

Let, $Q_m = [v_1 \, v_2 \ldots v_m]_{n\times m}$ $\quad Q_{m+1} = [v_1 \; v_2 \ldots v_{m+1}]_{n\times(m+1)}$

$$\overline{H}_m = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ h_{21} & h_{22} & & \vdots \\ & h_{32} & & \vdots \\ 0 & & & h_{m+1,m} \end{bmatrix}_{(m+1)\times m}$$

** Then, $$\underbrace{AQ_m}_{n\times m} = \underbrace{Q_{m+1}\overline{H}_m}_{n\times m}.$$

Comparing mth column,

$$Av_m = h_{1m}v_1 + h_{2m}v_2 + \cdots + h_{m+1,m}v_{m+1}$$

So, $$v_{m+1} = \frac{1}{h_{m+1,m}}\left(Av_m - \sum_{i=1}^{m} h_{im}v_i\right)$$

This iteration is known as Arnoldi iteration.

Steps :-

$$v_1 = \frac{b}{\|b\|_2}$$

$$v_2 = \frac{Av_1 - h_{11} v_1}{h_{21}}$$

Since, $\langle v_2, v_1 \rangle = 0 \Rightarrow h_{11} = \langle Av_1, v_1 \rangle = v_1^T A v_1 = r(A, v_1)$

Finally, $v = Av_1 - h_{11} v_1$ & $h_{21} = \|v\|_2$ and normalize

$$v_2 = \frac{v}{\|v\|_2}$$

Gram-Schmidt: $$v_n = \frac{x_n - \sum_{i=1}^{n-1} \langle x_n, v_i \rangle v_i}{r_{nn}}$$

Arnoldi $\equiv$ Gram-Schmidt

(1) Arnoldi iteration as projection onto Krylov subspaces :-

Consider the Krylov matrix

$$K_m = \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{m-1} b \end{bmatrix}_{n \times m}$$

Then $$AK_m = \begin{bmatrix} Ab & A^2 b & \cdots & A^m b \end{bmatrix}$$

$$= K_m \begin{bmatrix} e_2 & e_3 & \cdots & e_m & c \end{bmatrix}$$

where $c = K_m^{-1} A^m b$, assuming K_m is invertible.

So, $AK_m = K_m C_m$

$\Rightarrow K_m^{-1} A K_m = C_n$

So, A and C_n are similar.

Note that C_n is an Hessenberg matrix.

The matrix K_n is ill-conditioned as all its columns converse to the dominant eigen-vector of A.

The characteristic polynomial of C_n is

$$p(z) = z^n - \sum_{i=1}^{n} c_i z^{i-1}$$

where c_i are components of $-e$. The eigen-values of C_n are the roots of p.

$\Rightarrow$ Is Krylov subspace formulation is related to Gram-Schmidt?

Let, $K_n = Q_n R_n$.

Then, $K_n^{-1} A K_n = C_n$

$$\Rightarrow \quad R_n^{-1} Q_n^T A Q_n R_n = C_n$$

$$\Rightarrow \quad Q_n^T A Q_n = R_n C_n R_n^{-1}$$

$$= H_n.$$

The second approach is not a good option as:

$K_n c = A^n b$ is not well-posed.

*

$\Rightarrow$ If A is symmetric, Hessenberg matrices turn into tridiagonal. The method is called Lanczos iteration.

** $\Rightarrow$

$$A Q_m = Q_{m+1} \bar{H}_m$$

$\Rightarrow Q_m^T A Q_m = H_m$, removing the last row.

At each step m, we compute the eigen-values of H_m (By QR alogorithm). This provides estimates for m eigen-values.

Problems :-

1. Find the QR decomposition of $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}$ and then solve $Ax = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

2. Let, $A \in \mathbb{C}^{n \times n}$ & $b \in \mathbb{C}^n$ be arbitrary. Show that any $x \in K_m(A, b)$ is equal to $p(A)b$ for some polynomial p of degree $\leq m-1$

3. Show that K_m is an invariant subspace of A, i.e. $AK_m \subseteq K_m$ where $h_{m+1,m} = 0$.

4. If $h_{m+1,m} = 0$, then $K_m = K_{m+1} = K_{m+2} = \ldots$

*5. Let, A be a symmetric tridiagonal matrix that has no zero elements on its subdiagonal. Show that the matrix has distinct eigen-values.

6. Let, λ be an eigen-value of the orthogonal matrix Q. Show that $|\lambda| = 1$. What are the singular values of an orthogonal matrix?

*7. Let, $A = D + \rho u u^T$, where $D = \text{diag}(d_1, d_2, \ldots d_n)$ and u be a vector.

i) Prove that d_i is an eigen-value of A if $d_i = d_{i+1}$ or $u_i = 0$

ii) Prove that an eigen-vector corresponding to d_i is e_i if $u_i = 0$.

8. Let A^+ be the pseudo-inverse of $A_{m\times n}$. Prove

i) if $m=n$ and A is non-singular, $A^+ = A^{-1}$

ii) if $m>n$ & A has rank n, then $A^+ = (A^TA)^{-1}A^T$

iii) if $m<n$ and A has rank m, then $A^+ = A^T(AA^T)^{-1}$

9. $A = \begin{pmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{pmatrix}$. Transform the matrix A into a tridiagonal form using Householder reflection

10. Let A is an $m\times m$ matrix with SVD $A = U\Sigma V^T$. Find an eigen-value decomposition of $B = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}_{2m\times 2m}$.

1. $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}$ $\quad a_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ Gram-Schmidt

$$q_1 = \frac{a_1}{\|a_1\|_2} = \frac{1}{\sqrt5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad r_{11} = \|a_1\|_2 = \sqrt5$$

$$v_2 = a_2 - \langle a_2, q_1\rangle q_1$$

$$= \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \frac{1}{\sqrt5}\cdot\frac{1}{\sqrt5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -7/5 \\ 14/5 \\ 1 \end{pmatrix}$$

$$\therefore q_2 = \frac{v_2}{\|v_2\|_2} = \frac{10}{\sqrt{1080}}\begin{pmatrix} -7/5 \\ 14/5 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1080}}\begin{pmatrix} -14 \\ 28 \\ 10 \end{pmatrix}$$

$$= \frac{1}{\sqrt{270}}\begin{pmatrix} -7 \\ 14 \\ 5 \end{pmatrix}$$

$$= \frac{1}{\sqrt{30}}\begin{pmatrix} -7/3 \\ 14/3 \\ 5/3 \end{pmatrix}$$

$$r_{12} = \langle q_1, a_2\rangle = \frac{1}{\sqrt5}$$

$$r_{22} = \|v_2\|_2 = \frac{\sqrt{270}}{5} = \frac{3}{5}\sqrt{30}$$

$$v_3 = a_3 - \langle a_3, q_1\rangle q_1 - \langle a_3, q_2\rangle q_2$$

$$= \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} - 0 + \frac{15}{\sqrt{30}} \frac{1}{\sqrt{30}} \begin{pmatrix} -7/3 \\ 14/3 \\ 5/3 \end{pmatrix}$$

$$= \begin{pmatrix} -1/6 \\ 2/6 \\ -7/6 \end{pmatrix}$$

$$\therefore q_3 = \frac{v_3}{\|v_3\|_2} = \frac{6}{\sqrt{54}} \begin{pmatrix} -1/6 \\ 2/6 \\ -7/6 \end{pmatrix} = \frac{1}{\sqrt{54}} \begin{pmatrix} -1 \\ 2 \\ -7 \end{pmatrix}.$$

$$r_{13} = \langle q_1, a_3\rangle = 0, \quad r_{23} = \langle q_2, a_3\rangle = -\frac{15}{\sqrt{30}}$$

$$r_{33} = \frac{\sqrt{54}}{6} = \|v_3\|_2$$

So, $$Q = [q_1 \ q_2 \ q_3] = \begin{pmatrix} 2/\sqrt{5} & -7/\sqrt{270} & -1/\sqrt{54} \\ 1/\sqrt{5} & \frac{14}{\sqrt{270}} & \frac{2}{\sqrt{54}} \\ 0 & \frac{5}{\sqrt{270}} & -7/\sqrt{54} \end{pmatrix}$$

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \begin{pmatrix} \sqrt{5} & \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{3}{5}\sqrt{30} & -\frac{15}{\sqrt{30}} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2.2361 & 0.4472 & 0 \\ 0 & 3.2863 & -2.7386 \\ 0 & 0 & 1.2247 \end{pmatrix}}_{R} \Bigg| \underbrace{\begin{pmatrix} 0.8944 & -0.4260 & -0.1361 \\ 0.4472 & 0.8520 & 0.2722 \\ 0 & 0.3043 & -0.9526 \end{pmatrix}}_{Q}$$

$$Ax = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$\Rightarrow QRx = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \Rightarrow Rx = Q^T \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.6086 \\ -5.4433 \end{pmatrix}$$

$$\therefore x = \begin{pmatrix} 0.7778 \\ -3.8890 \\ -4.4446 \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}$$

<u>Householder</u>

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \sqrt{5}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_1 = \frac{v_1}{\|v_1\|_2} = \begin{pmatrix} -0.2298 \\ 0.9732 \\ 0 \end{pmatrix}$$

$$P_1 = I - 2u_1u_1^T = \begin{pmatrix} 0.8944 & 0.4472 & 0 \\ 0.4472 & -.8944 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_1A = \begin{pmatrix} 2.2361 & 0.4472 & 0 \\ 0 & -3.1305 & 2.2361 \\ 0 & 1 & -2 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -3.1305 \\ 1 \end{pmatrix} - \sqrt{3.1305^2+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6.4168 \\ 1 \end{pmatrix} \quad \therefore u_2 = \frac{v_2}{\|v_2\|_2} = \begin{pmatrix} -0.9881 \\ 0.1540 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{I - 2u_2u_2^T}} \\ 0 & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9526 & 0.3043 \\ 0 & 0.3043 & 0.9526 \end{pmatrix}$$

$$P_2P_1A = \begin{pmatrix} 2.2361 & 0.4472 & 0 \\ 0 & 3.2863 & -2.7386 \\ 0 & 0 & -1.2247 \end{pmatrix} = R$$

$$\therefore \quad A = (P_1P_2)R = QR$$

with
$$Q = P_1P_2 = \begin{pmatrix} 0.8944 & -0.4260 & 0.1361 \\ 0.4472 & 0.8520 & -0.2722 \\ 0 & 0.3043 & 0.9526 \end{pmatrix}$$

2. $A \in \mathbb{C}^{n\times n}, \ b \in \mathbb{C}^n$

$$x \in K_m(A,b) = \text{span}\{b, Ab, \dots \ A^{m-1}b\}$$

So,
$$x = c_0 b + c_1 Ab + c_2 A^2 b + \cdots + c_{m-1}A^{m-1}b$$
$$= (c_0 + c_1 A + c_2 A^2 + \cdots + c_{m-1}A^{m-1})\, b$$
$$= p_{m-1}(A)\, b$$

p is a polynomial of degree $\leq m-1$.

3. Arnoldi process:

$$A = QHQ^T.$$
$$\therefore \ AQ = QH.$$

$$A\,[v_1 \ v_2 \ \dots \ v_n] = [v_1 \dots \ v_n] \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1m} & \cdots & h_{1n} \\ h_{21} & h_{22} & & \vdots & & \vdots \\ 0 & h_{32} & & & & \\ \vdots & 0 & & h_{mm} & & \\ 0 & \cdot & & h_{m+1,m} & & \\ & & & 0 & h_{n,n-1} & h_{nn} \end{bmatrix}$$

Now, $h_{m+1,m} = 0$. Compare mth column:

$$Av_m = h_{1m}v_1 + \cdots + h_{mm}v_m \in \text{span}\{v_1 \dots, v_m\}$$
$$= K_m(A,b).$$

Now $\{v_1, v_2 \dots, v_m\}$ is an orthonormal basis of $K_m(A,b)$ using Gram-Schmidt.

So, $\underline{AK_m \subseteq K_m}$.

4.
$$AK_m = \text{span}\{Ab, A^2b, \dots, A^m b\} \subseteq K_m$$

$\Rightarrow A^m b \in K_m$ so that $K_{m+1} \subseteq K_m \Rightarrow K_{m+1} = K_m$

$$K_{m+2} = \text{span}\{b, Ab, \dots, A^{m+1}b\}$$

$A^m b = c_0 b + c_1 Ab + \cdots + c_{m-1}A^{m-1}b$

$\Rightarrow A^{m+1}b = c_0 Ab + \cdots + c_{m-1}A^m b \in K_m$ So, $A^{m+1}b \in K_m \Rightarrow K_{m+2} = K_m$

SVD

$$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}$$

Since $\text{rank}(A^TA) = \text{rank}(A) = 2$

S1 A^TA has zero as an eigen-value.

$\lambda_1 = 360$, $\lambda_2 = 90$, $\lambda_3 = 0$

So, the singular values of A are:

$\sigma_1 = \sqrt{360} = 6\sqrt{10}$, $\sigma_2 = \sqrt{90} = 3\sqrt{10}$,

$\sigma_3 = 0$

So, $$\Sigma = \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix}$$

S2 We need to find the eigen-vectors of A^TA.

$$V_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}.$$

$$\therefore V = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}.$$

S3 $$u_1 = \frac{Av_1}{\sigma_1} = \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}, \quad u_2 = \frac{Av_2}{\sigma_2} = \begin{pmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{pmatrix}$$

$$\therefore U = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix}$$

$$\therefore A = U \Sigma V^T.$$

6. $Qx = \lambda x$

$\Rightarrow \|Qx\|_2^2 = |\lambda|^2 \|x\|_2^2$

$\Rightarrow |\lambda|^2 \|x\|_2^2 = (Qx)^T (Qx)$

$= x^T Q^T Q x$

$= x^T x = \|x\|_2^2$

$\Rightarrow |\lambda|^2 = 1 \quad \Rightarrow \underline{|\lambda| = 1}$

$Q^T Q = I$. So, the eigenvalues of Q^TQ are all 1.

So, the singular values are 1.

10.

$A = U\Sigma V^T \Rightarrow AV = U\Sigma$

$A^T = V\Sigma U^T \Rightarrow A^T U = V\Sigma$

So, $\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} A^T U \\ AV \end{pmatrix} = \begin{pmatrix} V\Sigma \\ U\Sigma \end{pmatrix}$

Also, $\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} -V \\ U \end{pmatrix} = \begin{pmatrix} A^T U \\ -AV \end{pmatrix} = \begin{pmatrix} V\Sigma \\ -U\Sigma \end{pmatrix}$

So, $\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} V & -V \\ U & U \end{pmatrix} = \begin{pmatrix} V\Sigma & V\Sigma \\ U\Sigma & -U\Sigma \end{pmatrix}$

$= \begin{bmatrix} V & -V \\ U & U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$

Let, $P = \begin{bmatrix} V & -V \\ U & U \end{bmatrix}$

$P^T P = 2I$.

So, $\underline{P^{-1} B P = \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}}$

So, $\underline{Q = \frac{1}{\sqrt{2}} P}$