Numerical Techniques (MA2L007)

Syllabus:

Introduction

A.
- round-off error
- truncation error
- floating point
- convergence

Linear System

C.
- Gauss elimination
- Gauss-Jordan
- Gauss Seidel
- Banded matrices
- Skyline solver

Non-linear System

E.
- Newton Raphson
- Conjugate Gradient

ODE

B.
- Taylor
- Euler
- RK
- FDM

PDE

D. FDM : Laplace

: Heat

: Wave

Books :

1. Conter & Boor : Elementary Numerical Analysis

2. Dahlquist & Björck , Numerical Methods

Exam :-

CT	Mid Sem	LAB	End Sem
5%	30%	15%	50%
(7th Feb)	(29th Feb)	(13th April)	(23rd April)

Introduction :-

Numerical Methods provide procedures to obtain numerical solutions (approximation or estimate of the exact value) of problems.

$\Rightarrow$ An important aspect of studying numerical methods, is to study the error, during the procedure or in the final result.

Error = exact value − approximate value

Absolute Error = | Error |

$$\text{Relative Error} = \frac{|\text{Error}|}{|\text{Exact Value}|}$$

Why relative error ?

1. Let, an exact value $x^* = 0.001$, and approx. value is $x = 0.0016$. Then,

Absolute error $= |x - x^*| = 6 \times 10^{-4}$ (Very small)

But, the relative error $= \dfrac{|x - x^*|}{|x^*|} = 0.60$ (i.e. 60%)

indicating that the error margin is indeed large.

2. Let, $x^* = 12000$, $x = 12005$

So, AE $= 5$

but, RE $= \dfrac{5}{12000} = 4.17 \times 10^{-4}$ (error margin is small).

Types of error :-

1. Initial error / Inherent error :-

These errors are involved in the statement of the problem itself, before its solution.

Exm: gravitational acceleration g. (measurement error)

$$y' = y, \quad y(0) = 1 \quad \leftrightarrow \quad y(0) = 1.01$$
$$y(1) = ? \qquad\qquad y(1) = ?$$

2. Truncation error :-

We know

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots \infty$$

(Gregory - Leibnitz)

$$\pi = 3 + \frac{4}{2.3.4} - \frac{4}{4.5.6} + \frac{4}{6.7.8} - \frac{4}{8.9.10} + \cdots \infty$$

(Nilakantha)

To calculate the accurate value of the functions at a point x or to find the value of π, we need to add more and more terms, ∴ which is difficult.

An error gets introduced for not considering the remaining terms or for truncating the series after finite terms.
This leads to truncation error.

3. Round-off error :-

$\frac{1}{3} = 0.3333\ldots$ $\qquad$ $\frac{22}{7} = 3.14285\ldots$

$$\left(\frac{223}{71} < \pi < \frac{22}{7}\right)$$

$$\pi \simeq \frac{355}{113}$$

$\pi = 3.1415926535897\ldots$

How I wish I could calculate pi.

$\sqrt{2} = 1.41421356\ldots$, $e = 2.718281828 45\ldots$

For practical calculation, we are forced to take a few number of digits from their expression, thus an error named round-off error gets involved.

Significant Digits & Decimal Points :-

The total no of digits to the right of the point (·) are known as 'decimal points', where as significant digits start from the first non-zero number.

Exm : — 0.7452 and 0.007452 both have 4 s.d, but 4 and 6 d.p respectively.

— 0.4030 has 4 s.d as well as 4 d.p.

— 0.0091 is 0.01 (to 2 d.p) but 0.0091 (to 2 s.d)

Exr : Round-off 37.897456 correct upto 5 s.d & 5 d.p

37.897 (upto 5 s.d) , 37.89746 (upto 5 d.p)

Exr: Round-off 6.002500 correct upto 4 s.d

6.002500 becomes 6.002 (4 s.d)

Since the 4th place is even (=2), it is unaltered.

Exr: Round-off 5.001500 upto 4 s.d.

5.001500 becomes 5.002, as 4th place is odd.

Exr: Round-off 0.000000123456 correct upto 4 s.d.

Floating Point Representation :-

A floating point representation of a number is given by:

$$\pm M \times \beta^{e}$$

where M is called the <u>mantissa</u> and e is an integer, called the <u>exponent</u>. Such a floating pt. number is called normalized.

$$\left(\frac{1}{\beta} \leq M < 1\right)$$

Most computers use $\beta = 2$ or 16.

Exm:

$$\left.\begin{aligned} 5431 &= 0.5431 \times 10^{4} \\ -1.23 &= -0.123 \times 10^{1} \\ .0056 &= 0.56 \times 10^{-2} \end{aligned}\right\} \begin{array}{c} 10.\text{ base.} \\ (\beta = 10) \end{array}$$

The digits in mantissa M is called significant digits.

$\Rightarrow$ An n-digit floating point number has the form:

$$x = \pm (\cdot d_1 \ldots d_n)_{\beta} \times \beta^{e}$$

where $M = (\cdot d_1 \ldots d_n)_{\beta}$ is called a β-fraction.

Exm: $(\cdot 101)_2 \times 2^4 = (2^{-1} + 2^{-3}) \times 2^4 = 2 + 2^3$
$= 10.$

$$(\cdot d_1 \ldots, d_n)_\beta = \sum_{k=1}^{n} d_k \cdot \beta^{-k}$$

A floating pt no. in normalized if $d_1 \neq 0$ or

$$d_1 = 0 = d_2 = \ldots = d_n$$

Books :-

* *1 Numerical Methods - Problem & solns , Jain, Iyengar
* *2. Applied Numerical Analysis , Curtis Gerald & P. Wheatley
* 3. Numerical Methods, B.S. Grewal
* 4. Numerical Methods for Scientist & Engineers, Hamming

Notation & Existence of Solutions :-

First order differential equation:

$$\frac{dy}{dt} = f(t, y)$$

Exm: $y' = f(t, y) = y$

$\Rightarrow$ $y = e^t$ is a soln.

In fact $y = ce^t$ are infinite solns of the eqn.

To obtain a unique sol^n, we prescribe an initial condition $y(0) = 1$.

In general a diff. eq^n of order p has the form:

$$F(t, y, y', \dots, y^{(p)}) = 0.$$

The general sol^n will contain p constants of integration. By requiring p initial or boundary cond^{n}s, a particular sol^n is selected from the family of sol^{n}s.

Initial cond^n :- Conditions are for same point $t = t_0$.

EXM. $y' = y^2$, $y(0) = 1$.

Boundary Cond^n :- cond^n are spread over diff. pts. t.

EXM: $y'' = y + t$, $y(0) = 1$, $y(1) = 2$

Theoretically, they have no diff. but to determine a particular sol^n.

But, numerically, there is a big difference. IVP are easier to solve numerically.

Lipschitz but not $\frac{\partial f}{\partial y}$ exists.

$$f(t, y) = t^2 |y|, \quad |t| \le 1, |y| \le 1$$

$\frac{\partial f}{\partial y}(t, 0)$ does not exists.

Existence & Uniqueness Theorem :-

$$y' = f(t, y) \quad , \quad y(a_0) = y_0 \qquad \cdots (*)$$

Let, $f(t, y)$ be continuous for $-\infty < a \leq t \leq b < \infty$ & $-\infty < y < \infty$ and suppose $\exists\ L > 0$ s.t.

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2| \qquad \cdots (**)$$

for all $t \in [a, b]$ & $y_1, y_2 \in \mathbb{R}$, (Lipschitz condn).

Then, for any given $y_0 \in \mathbb{R}$, $\exists$ a unique fn $y(t)$ such that

1) $y \in C^1[a, b]$.

ii) $y' = f(t, y) \quad \forall\, t \in [a, b]$

iii) $y(a) = y_0$.

This is called Picard - Lindelöf Theorem.

$\Rightarrow$ The eqn $(**)$ can also be replaced by the condn : $\dfrac{\partial f}{\partial y}$ exists.

Exm : $y' = |y|$, $y(a) = y_0$.

Solution : $y(t) = \begin{cases} y_0\, e^{t-a} & , \ y_0 > 0 \\ y_0\, e^{a-t} & , \ y_0 < 0 . \end{cases}$

Exm : $y' = \sqrt{|y|}$, $y(2) = -1$.

f is continuous but does not satisfy Lipschitz condn.

$y_1, y_2 > 0$, let, $y_2 \to y_1$ then,

$$\frac{\left|\sqrt{|y_1|} - \sqrt{|y_2|}\right|}{|y_1 - y_2|} \to \frac{1}{2\sqrt{y_1}}$$ which is unbounded for

We can construct solution as:

$$y(t) = \begin{cases} -\dfrac{(t-4)^2}{4}, & 2 \le t < 4 \\ 0, & 4 \le t \le a \\ \dfrac{(t-a)^2}{4}, & t > a \end{cases}$$

for $a > 0$

The sol$\underline{n}$ is not unique, depends on arbitrary a.

However, it is often impossible to solve a diff. eq$\underline{n}$ analytically:

$$y' = t^2 + y^2, \quad y(0) = 1.$$

(It has a unique solution !)

Also, even when an analytical sol$\underline{n}$ exists, it may be such an expression that it is not useful.

Exm. $\quad y'' = t^2 y + t + 1 - \frac{1}{t}$.

(Solution involves Bessel functions)

A number of numerical methods are available to solve / approximate the sol$\underline{n}$ of:

$$\frac{dy}{dt} = f(t, y); \quad y(a) = y_0.$$

Two types:

1. Power Series form
   - Picard
   - Taylor

   } Classical methods

2. Set of values of t & y:
   - Euler
   - Runge - Kutta
   - Adam - Bashforth.