Solving Non-linear System :-

$f : [a,b] \to \mathbb{R}$ function.

Aim: To find $c \in [a,b]$ such that $f(c) = 0$.

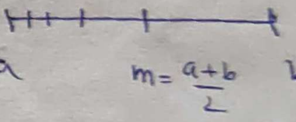
Result 1: (Intermediate Value Theorem)

If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a).f(b) \leq 0$, then $\exists\ c \in [a,b]$ s.t.

$$f(c) = 0.$$

Bisection Method:

1. Let, $f(a).f(b) < 0$.



2. $m = \dfrac{a+b}{2}$

   If $f(m) = 0 \Rightarrow c = m$

   If not, either $f(a).f(m) < 0$ or $f(m).f(b) < 0$

   $\downarrow$ $\qquad\qquad$ $\downarrow$

   $a_1 = a,\ b_1 = m$ or $a_1 = m,\ b_1 = b$

3. $m_i = \dfrac{a_i + b_i}{2}$ and repeat - 2.

4. $a_i \to c,\ b_i \to c$ as $i \to \infty$

Defn: f has a simple zero at c if $f(c) = 0$, but $f'(c) \neq 0$.

Newton's Method (Newton - Raphson)

Let, f is a continuously differentiable function.

At a certain stage we reach x_k, which is close to its root.

Let, $f(x_k + \Delta x) = 0$

$$\Rightarrow f(x_k) + \Delta x\, f'(x_k) + \frac{(\Delta x)^2}{2!} f''(x_k) + \dots = 0$$

Ignoring higher power of (Δx), we get,

$$f(x_k) + \Delta x\, f'(x_k) = 0$$

$$\Rightarrow \Delta x = -\frac{f(x_k)}{f'(x_k)}, \quad f'(x_k) \neq 0$$

So, the next approximation is:

$$x_{k+1} = x_k + \Delta x$$

$$= x_k - \frac{f(x_k)}{f'(x_k)}, \quad f'(x_k) \neq 0.$$

$$k = 0, 1, \dots$$

Here, x_0 is an initial guess.

$\Rightarrow$ Let c be the root of f. Then the error at the kth approximation is:

$$e_k = x_k - c.$$

If $\exists$ a number p and a constant M such that

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|^p} = M$$

then p is called the order of convergence of the iteration $\{x_k\}$.

Exm: $f(x) = x^2 - 2 = 0$. The root of $f(x) = 0$ yields the value $\sqrt{2}$.

Newton-Raphson iteration:

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k}, \quad x_k \neq 0.$$

$$x_{k+1} = \frac{x_k^2 + 2}{2x_k}, \quad k = 0, 1, \ldots$$

iter	x_k	$f(x_k)$
0	1	-1
1	$\frac{3}{2}$	$\frac{1}{4}$
2	1.4167	0.0069
3	1.4142	6.17×10^{-6}
4	1.4142	—

Convergence of Newton's Method :—

Let, f has a continuous 2nd derivative. Then by Taylor's theorem. $[f \in C^1$ and f'' exists$]$ ← condition on f ①

$$f(\alpha) = f(x_k) + f'(x_k)(\alpha - x_k) + \frac{f''(c_k)}{2!}(\alpha - x_k)^2,$$

$c_k \in (\alpha, x_k)$

Since, α is a root of f, $f(\alpha) = 0$

$$\Rightarrow f(x_k) + f'(x_k)(\alpha - x_k) + \frac{1}{2} f''(c_k)(\alpha - x_k)^2 = 0$$

Dividing by $f'(x_k)$, we get,

$$\alpha - \left\{x_k - \frac{f(x_k)}{f'(x_k)}\right\} + \frac{1}{2}\frac{f''(c_k)}{f'(x_k)}(\alpha - x_k)^2 = 0$$

$$\Rightarrow \quad \alpha - x_{k+1} = -\frac{f''(c_k)}{2f'(x_k)}(\alpha - x_k)^2$$

$$\Rightarrow \quad e_{k+1} = \frac{f''(c_k)}{2f'(x_k)} e_k^2$$

$$\frac{|e_{k+1}|}{e_k^2} = \frac{f''(c_k)}{2f'(x_k)} \quad \text{---} (*)$$

If $M = \sup_{x \in I} \frac{1}{2}\left|\frac{f''(x)}{f'(x)}\right|$, then

$$|e_{k+1}| \leq M e_k^2$$

This shows that the rate of convergence is quadratic.

Fixed point iteration :-

α is said to be a fixed point of g if

$$g(\alpha) = \alpha.$$

$\Rightarrow$ $f(x) = 0$ can be rewritten as

$$g(x) = x.$$

Exm $x^2 - x - 1 = 0$ can be written as:

i) $x = x^2 - 1$

ii) $x = \sqrt{x+1}$

iii) $x = 1 + \frac{1}{x}$

Each such $g(x)$ is called an iteration function.

Fixed point iteration

S1 Choose an initial guess x_0

S2. Calculate the sequence $\{x_n\}$ by

$$x_{n+1} = g(x_n), \quad n \geq 0.$$

$\Rightarrow$ Not all initial point x_0 leads to the iteration $\{x_n\}_n$ [EXM: $g(x) = -\sqrt{x}$]

$\Rightarrow$ If the sequence $\{x_n\}_n$ converges to α. then α is a fixed point of g; $g(\alpha) = \alpha$.

Assumption :- 1. There is an interval / domain I such that for $x \in I$, $g(x) \in I$.

2. Let, $x_n \to \alpha$ then $x_{n+1} = g(x_n)$ leads to

$$\alpha = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n) = g(\alpha)$$

provided g is continuous.

g is continuous.

3. $g(x)$ is differentiable on I and $\exists$ a constant $K<1$, such that

$$|g'(x)| \leq K < 1 \quad \forall x \in I.$$

Theorem :- Assume $g \in C'([a,b])$ with $|g'(x)| \leq K$ for all $x \in I$ with $K<1$. Also, $g: [a,b] \to [a,b]$

Then,

i) g has a unique fixed point α.

ii) $x_{n+1} = g(x_n)$, starting with any $x_0 \in I$ converges to α.

iii) $$|x_{n+1} - \alpha| \leq K^n |x_0 - \alpha| \leq \frac{K^n}{1-K} |x_1 - x_0|$$

and $$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

Proof :- ii) Existence of a fixed point α is proved by applying IVP on $F(x) = g(x) - x$.

Now, $|x_{n+1} - \alpha|$

$$= |g(x_n) - g(\alpha)|$$

$$= |g'(c_n)| \, |x_n - \alpha| \quad \text{[By L.M.V.T]} \quad c_n \in (x_n, \alpha)$$

$$|x_{n+1} - \alpha| \leq K |x_n - \alpha| \quad \ldots (*)$$

So that, $$|x_{n+1} - \alpha| \leq K^n |x_0 - \alpha|$$

As $K<1$, $\lim_{n \to \infty} K^n \to 0$ so that

$$\lim_{n \to \infty} x_n = \alpha.$$

i) Let β is another fixed pt. of g.

Take $x_0 = \beta$, then $x_1 = g(x_0) = \beta$

So, $(*) \Rightarrow |x_1 - \alpha| \leq K |x_0 - \alpha|$

$$\Rightarrow |\beta - \alpha| \leq K |\beta - \alpha|$$

Since $K<1$, $|\beta - \alpha| = 0 \Rightarrow \beta = \alpha$.

iii)

$$|\alpha - x_0| = |x_0 - x_1 + x_1 - \alpha|$$
$$\leq |x_0 - x_1| + |x_1 - \alpha|$$
$$\leq |x_0 - x_1| + k|x_0 - \alpha|$$
$$\Rightarrow \quad |\alpha - x_0| \leq \frac{1}{1-k}|x_1 - x_0|$$

We have $\quad |x_{n+1} - \alpha| \leq k^n |x_0 - \alpha|$
$$\leq \frac{k^n}{1-k}|x_1 - x_0|$$

Moreover,

$$\alpha - x_{n+1} = g(\alpha) - g(x_n) = g'(c_n)(\alpha - x_n)$$
$$\Rightarrow \quad \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(c_n) \qquad \alpha < c_n < x_n$$

Now, as $n \to \infty$, $x_n \to \alpha$, so $c_n \to \alpha$.

i.e. $$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to\infty} g'(c_n)$$
$$= g'(\alpha) \quad [\because g' \text{ is continuous}]$$

~~Exm~~ -

Note: So, by (iii) $x_n \to \alpha$ linearly.

Exm: $\quad f(x) = \sin x + x^2 - 1 = 0, \quad x \in [0,1]$

1) $g_1(x) = \sin^{-1}(1 - x^2), \quad |g_1'(x)| = \left|\frac{-2}{\sqrt{2-x^2}}\right| > 1$

2) $g_2(x) = -\sqrt{1 - \sin x} \qquad x_n$ does not exists for $x_0 > 0$

✓ 3) $g_3(x) = \sqrt{1 - \sin x} \qquad |g_3'(x)| = \left|\frac{-\cos x}{\sqrt{1-\sin x}}\right| \leq \frac{1}{\sqrt{2}} < 1$

Fixed point iteration $\longleftrightarrow$ Newton - Raphson

Define $g(x) = x - \dfrac{f(x)}{f'(x)}$, $x \in I$

Finding root of $f(x) = 0$ is equivalent to finding fixed point of g, $g(\alpha) = \alpha$.

Result / Theorem :-

Let, $f \in C^2([a,b])$ and $\exists\, \alpha \in [a,b]$ such that $f(\alpha) = 0$. If $f'(\alpha) \neq 0$, then $\exists$ a $\delta > 0$ s.t. Then seq:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1 \ldots$$

converges to α for an initial guess

$$x_0 \in [\alpha - \delta, \alpha + \delta]$$

Moreover, $$\lim_{n \to \alpha} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^2} = \frac{|f''(\alpha)|}{2|f'(\alpha)|}$$

Proof:

$$g'(x) = \frac{f(x) f''(x)}{(f'(x))^2} \quad x \in I.$$

Since $f(\alpha) = 0 \Rightarrow g'(\alpha) = 0$

As g' is continuous, $\exists\, \delta > 0$ s.t.

$$|g'(x)| < 1, \quad x \in (\alpha - \delta, \alpha + \delta).$$

Choose $x_0 \in (\alpha - \delta, \alpha + \delta)$ and $\delta > 0$ such that

$$\left| \frac{f(x) f''(x)}{(f'(x))^2} \right| < 1 \quad \forall x \in (\alpha - \delta, \alpha + \delta)$$

Taylor's thm.

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(c_n)$$

$$c_n \in (\alpha, x_n)$$

Now, $|g'(x)| < 1$ suffices $x_n \to \alpha$ as $n \to \infty$.

Also, as $n \to \infty$, $c_n \to \alpha$. This leads to:

$$\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{(x_n - \alpha)^2} = \frac{|f''(\alpha)|}{2|f'(\alpha)|}$$

H.W. Start with $x_0 = -2.4$ and use Newton-Raphson iteration to find the root $\alpha = -2$ of the polynomial $f(x) = x^3 - 3x + 2$. Verify

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^2} \simeq \frac{2}{3}$$

H.W. Let, $f(x) = \cos x$. For the root $\alpha = \frac{\pi}{2}$, if one starts from $x_0 = 3$, what it leads to. Explain.

System of non-linear equations :-

Instead of a single non-linear eqn, we have a system $f : \mathbb{R}^k \to \mathbb{R}^k$ of non-linear equation. To find a root of $f(\bar{\alpha}) = \bar{0}$, Newton-Raphson iteration is:

$$\bar{x}_{k+1} = \bar{x}_k - J_f(\bar{x}_k)^{-1} f(\bar{x}_k), \quad k \geq 0$$

where J_f is the jacobian matrix of f.

Gradient : $f : \mathbb{R}^K \to \mathbb{R}$ be a function. The gradient of f, ∇f is defined by

$$\nabla f(\bar{x}) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_K}\right)_{1\times K}.$$

Jacobian :- Let, $f : \mathbb{R}^k \to \mathbb{R}^m$ is a function.

Let, $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$. The Jacobian of f is an $m\times K$ matrix defined by

$$J_f(\bar{x}) = \left(\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ \cdots \ \frac{\partial f}{\partial x_K}\right)$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_K} \\ \frac{\partial f_2}{\partial x_1} & & \vdots \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_K} \end{bmatrix}_{m\times K}.$$

Exm :-

$$f_1 = 3x_1^2 + 4x_2^2 - 1 = 0$$

$$f_2 = x_2^3 - 8x_1^3 - 1 = 0, \quad \bar{x}_0 = \begin{pmatrix} -0.5 \\ 0.25 \end{pmatrix}$$

$$J_f(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} 6x_1 & 8x_2 \\ -24x_1^2 & 3x_2^2 \end{pmatrix}$$

$$J_f^{-1} = \frac{1}{192x_1 + 18x_2} \begin{pmatrix} \frac{3x_2}{x_1} & -\frac{8}{x_1} \\ \frac{24x_1}{x_2} & \frac{6}{x_2} \end{pmatrix}$$

Newton iteration:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{K+1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{K} - J_f^{-1}(x_1^k, x_2^k)\, f(x_1^k, x_2^k),$$

$K \geq 0$.

i.e
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{1} = \begin{pmatrix} -0.5 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.0164 & -0.1749 \\ 0.5246 & -0.2623 \end{pmatrix} \begin{pmatrix} 0 \\ 0.0156 \end{pmatrix}$$

$$= \begin{pmatrix} -0.4973 \\ 0.2541 \end{pmatrix}$$ and so on.

H.W. Let, $\{x_n\}$ be the iterative seq$^{\underline{n}}$ generated by Newton's method for finding the root of $e^{-ax} = x$, $0 < a \leq 1$. If x^* is the exact root of the eq$^{\underline{n}}$ and $x_0 > 0$, then show that

$$|x^* - x_{n+1}| \leq \frac{1}{2}(x^* - x_n)^2.$$