

10.2Calculus of Parametric curves: -

$$(x, t) = (f(t), g(t))$$

~~dx~~

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if } \frac{dx}{dt} \neq 0.$$

#

$$x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \sin \theta}{r(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta}$$

$$\theta = \pi/3$$

$$\frac{dy}{dx} = \sqrt{3}$$

$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \quad y = r\left(1 - \frac{1}{2}\right) = r/2$$

$$y - r/2 = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{\sqrt{3}r}{2}\right)$$

Horizontal: $\frac{dy}{dx} = 0 \Rightarrow \sin \theta = 0 \quad \& \quad 1 - \cos \theta \neq 0$

$$\Rightarrow \theta = (2n-1)\pi$$

$$\lim_{\theta \rightarrow 2n\pi+} \frac{dy}{dx} = \lim_{\theta \rightarrow 2n\pi+} \frac{\sin \theta}{1 - \cos \theta} = \lim_{\theta \rightarrow 2n\pi+} \frac{\cos \theta}{\sin \theta} = \infty.$$

$$\text{ll. by } \lim_{\theta \rightarrow 2n\pi-} \frac{dy}{dx} = -\infty$$

The tangent is vertical.

(2)

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{dy}{dt} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) / \frac{dx}{dt}$$

Area: -

$$A = \int_a^b y dx = \int_{\alpha}^{\beta} g(t) f'(t) dt.$$

$x = a \cos \theta, y = b \sin \theta, 0 \leq \theta \leq 2\pi.$

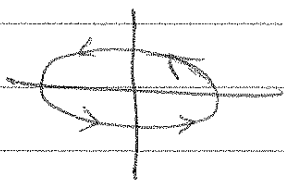
$$A = - \int_0^{2\pi} b \sin \theta a \sin \theta d\theta$$

$$= -ab \int_0^{2\pi} \sin^2 \theta d\theta = -4ab \int_0^{\pi/2} \sin^2 \theta d\theta$$

$$= -2ab \int_0^{\pi/2} (1 - \cos 2\theta) d\theta$$

$$= -2ab \left(\frac{\pi}{2} \right) = -\pi ab$$

due to orientation.



$$\underline{A = \pi ab}$$

③

Arc Length

$$S = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

$$x = t, \quad y = f(t) (= f(x))$$

$$S = \int_{\alpha}^{\beta} \sqrt{1 + [f'(x)]^2} dx$$

$x = r \cos t, \quad y = r \sin t.$

$$\frac{dx}{dt} = -r \sin t, \quad \frac{dy}{dt} = r \cos t.$$

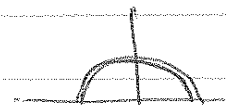
$$S = \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2 t)} dt = r \int_0^{2\pi} dt = \underline{2\pi r}$$

Surface area:- rotated about x/y axis.

$$\int 2\pi x ds \quad \text{or} \quad \int 2\pi y ds.$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Sphere: $x = r \cos t, \quad y = r \sin t, \quad t \in [0, \pi]$



$$\begin{aligned} S &= \int_0^{\pi} 2\pi r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)} dt \\ &= 2\pi r^2 \int_0^{\pi} \sin t dt = \underline{4\pi r^2} \end{aligned}$$

④

~~$x = 8 \tan\left(\frac{t}{24}\right), y = 12 \sec\left(\frac{t}{24}\right)$~~

$x = 4 - t^2, y = t^3 - 27t.$

$$\frac{dx}{dt} = -2t, \quad \frac{dy}{dt} = 3t^2 - 27$$

$$\frac{dy}{dx} = \frac{3t^2 - 27}{-2t}$$

Horizontal: $\frac{dy}{dx} = 0 \Rightarrow 3(t^2 - 9) = 0$
 $\Rightarrow t = \pm 3.$

$$x = -5, y = 27 - 81, -27 + 81$$

$$(-5, -54), (5, 54)$$

Vertical: $\frac{dy}{dx} \rightarrow \infty$ for $t = 0.$

$$x = 4, y = 0: (4, 0)$$

$x = 8 \sin t, y = 2 \sin(t + 5 \sin t)$

$$\frac{dx}{dt} = 8 \cos t, \quad \frac{dy}{dt} = 2 \cos(t + 5 \sin t) \cdot (1 + 5 \cos t)$$

$$\therefore \frac{dy}{dx} = \frac{(1 + 5 \cos t) (\cos(t + 5 \sin t))}{4 \cos t}$$

$$(x, y) = (0, 0) \Rightarrow t = 0, \pi.$$

$$\left. \frac{dy}{dx} \right|_{t=0} = \frac{6}{4}, \quad \left. \frac{dy}{dx} \right|_{t=\pi} = -1$$

$$\text{and } y-0 = \frac{6}{4}(x-0) \text{ and}$$

$$y-0 = -(x-0)$$

$$\# \quad x = 4 \cos(4t), \quad y = 16t + 4 \sin(4t), \quad 0 \leq t \leq \pi/4$$

$$\text{Arc length, } s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{\pi/4} \sqrt{2.16^2 + 2.16^2 \cos(4t)} dt = 16$$