(1)

7.4 Integration by partial fractions:

Rational fraction $\left(\frac{P(x)}{Q(x)}\right)$ to simpler fractions!

$$\int \frac{x^3+x+1}{x-1}\,dx. \qquad \int \frac{x+5}{x^2+x-2}\,dx.$$

$$
\begin{array}{r|l}
x-1 & x^3+x+1 \quad (x^2+x+2 \\
 & \underline{-x^3 + x^2} \\
 & x^2+x+1 \\
 & \underline{-x^2 + x} \\
 & 2x+1 \\
 & \underline{-2x + 2} \\
 & 3
\end{array}
$$

So, $\dfrac{x^3+x+1}{x-1} = x^2+x+2 + \dfrac{3}{x-1}$

$$\int \frac{x^3+x+1}{x-1}\,dx = \int\left(x^2+x+2+\frac{3}{x-1}\right)dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 3\ln|x-1| + C$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{x+5}{x^2+2x-x-2}\,dx$$

$$= \int \frac{x+5}{(x+2)(x-1)}\,dx$$

$$\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \Rightarrow \begin{array}{l} A+B=1 \\ -A+2B=5 \end{array} \Rightarrow \begin{array}{l} B=2 \\ A=-1 \end{array}$$

②

i) Factor the denominator

ii) Write as

$$\frac{P(x)}{Q(x)} = \frac{A}{p_1x+q_1} + \frac{B}{p_2x+q_2} + \cdots + \frac{C}{p_3x+q_3}.$$

Find A, B, C.

iii) Integrand individual factor!

\# $\displaystyle\int \frac{dx}{x^2-a^2}$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a}$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

$$= \underline{\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C}.$$

\# $$\frac{x^3}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

NOT always possible to get linear factor!

$$\frac{x}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}.$$

(3)

35 $\displaystyle\int \frac{dx}{x(x^2+4)^2}$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$\Rightarrow A = \frac{1}{16}, \quad B = -\frac{1}{16}, \quad C = 0 = E$$

$$D = -\frac{1}{4}.$$

$$\int \frac{dx}{x(x^2+4)^2} = \frac{1}{16}\int \frac{dx}{x} - \frac{1}{16}\int \frac{x}{x^2+4}\,dx - \frac{1}{4}\int \frac{x}{(x^2+4)^2}\,dx$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+4|$$

$4+x^2 = u$

$$-\frac{1}{8}\int \frac{du}{u^2}.$$

$2x\,dx = du$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+4| + \frac{1}{8u} + C$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+4| + \frac{1}{8(x^2+4)} + C.$$

31 $\displaystyle\int \frac{1}{x^3-1}\,dx$

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\begin{cases} A+B=0 \\ -B+C+A=0 \\ A-C=1 \end{cases}$$

47 $$\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$$ Let, $e^x = u$

$e^x dx = du$

$$= \int \frac{u}{u^2+3u+2}\,du$$

$$= \int \frac{u}{(u+2)(u+1)}\,du$$

$$\frac{u}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

$A = 2,\ B = -1$

$$= \int \frac{2}{u+2}\,du - \int \frac{du}{u+1}$$

$$= 2\ln|e^x+2| - \ln|e^x+1| + c \quad = \ln\left(\frac{(e^x+2)^2}{e^x+1}\right) + c.$$

50 Take $e^x = u$.

51. $$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}\,dx}{1+e^{-x}}$$ Let, $e^{-x} = u$

$-e^{-x}dx = du$

$$= -\int \frac{du}{1+u}$$

$$= -\ln|1+u| + c$$

$$= -\ln\left|1+\frac{1}{e^x}\right| + c.$$

$$= -\ln|1+e^x| + \ln e^x + c$$

$$= x - \ln(1+e^x) + c$$

⑤

53

$$\int \ln(x^2-x+2)\,dx.$$

$$= x\ln(x^2-x+2) - \int \frac{(2x-1)\,x}{x^2-x+2}\,dx.$$

$$= x\ln(x^2-x+2) - \int \left[2 + \frac{x-4}{x^2-x+2}\right] dx.$$

$$= x\ln(x^2-x+2) - 2x - \int \frac{x-4}{x^2-x+2}\,dx.$$

$$\frac{x-4}{x^2-x+2} = \frac{A(2x-1)}{x^2-x+2} + \frac{B}{x^2-x+2}$$

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}$$

$$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln|x^2-x+2| + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{7}{4}}.$$

$x-\frac{1}{2} = \sqrt{\frac{7}{4}}\tan\theta$

$dx = \sqrt{\frac{7}{4}}\sec^2\theta\,d\theta$

$$= \left(x-\frac{1}{2}\right)\ln(x^2-x+2) - 2x + \frac{7}{2}\cdot\frac{2}{\sqrt{7}}\,\theta + C$$

$$= \left(x-\frac{1}{2}\right)\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C$$

63

$$\int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\,dx.$$

Let, $u = \cos x$

$du = -\sin x\,dx.$

$$= \int_0^{\pi/2} \frac{2\sin x\cos x}{2+\cos x}\,dx$$

$$= -\int_1^0 \frac{2u}{u+2}\,du = 2\int_0^1 du - 4\int_0^1 \frac{du}{u+2}$$

$$= 2 - 4\ln(u+2)\Big|_0^1 = 2 + 4\ln\frac{2}{3}$$