10.4 Areas of Polar curves :-

$$r = f(\theta)$$

Formula: $A = \int_a^b \frac{1}{2} r^2 \, d\theta.$

$r^2 = -8r\cos\theta$

$\Rightarrow x^2 + y^2 = -8x \Rightarrow (x+4)^2 + y^2 = 16$

Circle: center: $(-4, 0)$, radius: 4.

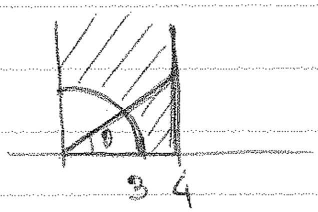
$r = 2a\sin\theta \Rightarrow r = 2a\, y/r$

$\Rightarrow r^2 = 2ay \Rightarrow x^2 + y^2 - 2ay + a^2 = a^2$

$\Rightarrow x^2 + (y-a)^2 = a^2$

Center: $(0, a)$, radius $= a$

#



$0 \le \theta \le \pi/2$

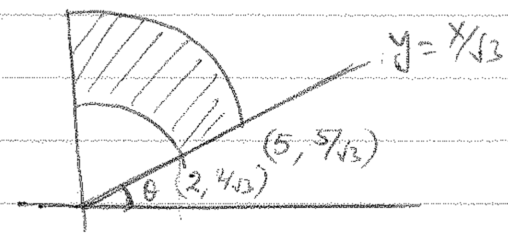
$r \ge 3$

$\frac{4}{r} = \cos\theta \Rightarrow r = 4/\cos\theta$

So, $3 \le r \le 4/\cos\theta$

#

$$\sqrt{2^2+(2/\sqrt{3})^2} \le r \le \sqrt{5^2+(5/\sqrt{3})^2}$$

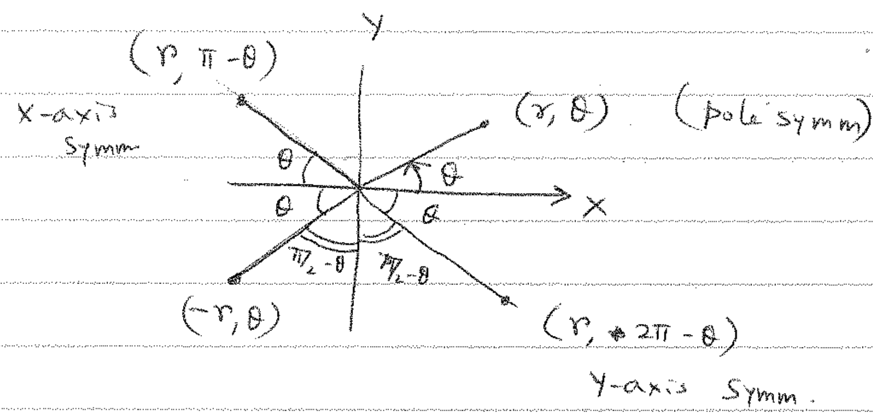


θ &

$$\tan\theta = \frac{2/\sqrt{3}}{2} = 1/\sqrt{3}$$

$\Rightarrow \theta = \pi/6. \qquad \Rightarrow \underline{\pi/6 \le \theta \le \pi/2}$

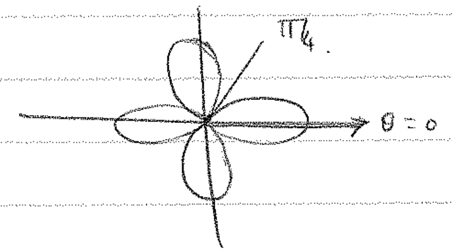
Symmetry w.r.to x, y-axis. $(-r,\theta)$ $\underline{0 \le \theta \le 2\pi}$, $\underline{r>0}$



$(r,\theta) \rightarrow$ symm. w.r.to x & y axis.

Area :- $r = 3\cos 2\theta$ Four-leaved Rose

One leaf

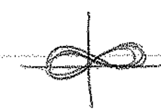


$$A = 2 \cdot \int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= 2 \cdot \int_0^{\pi/4} \frac{1}{2} \, 9\cos^2 2\theta \, d\theta$$

$$= \frac{9}{2} \int_0^{\pi/4} (1 + \cos 4\theta) \, d\theta = \frac{9}{2} \left[\frac{\pi}{4} \right] = \frac{9\pi}{8}.$$

\# $r^2 = 100\cos(2\theta)$ Area inside lemniscate.

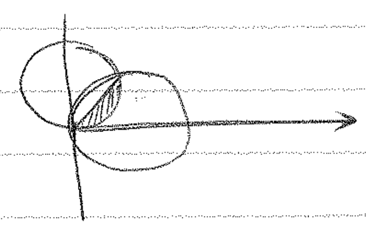


$$A = 2 \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, d\theta = 2 \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} 100\cos(2\theta) \, d\theta$$

$$= 2 \int_0^{\pi/4} 100 \cos(2\theta) \, d\theta$$

$$= 200 \, \frac{1}{2} = 100.$$

\# $r = 6\cos\theta$, $r = 6\sin\theta$.

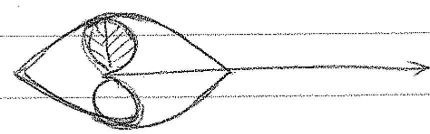


$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/4} r^2 \, d\theta$$

$$= \int_0^{\pi/4} (6\sin\theta)^2 \, d\theta = 18 \cdot \int_0^{\pi/4} (1 - \cos 2\theta) \, d\theta$$

$$= 18 \left[\pi/4 - \frac{1}{2} \right]$$

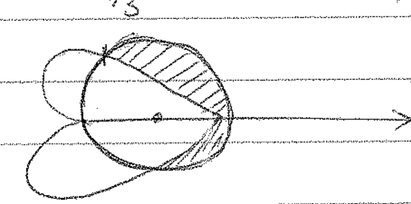
\# $r = 8(1+\cos\theta)$ $\quad r = 8(1-\cos\theta)$



$$A = 2\int_0^{\pi/2} \frac{1}{2} 8^2(1-\cos\theta)^2\, d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2} 8^2(1+\cos\theta)^2\, d\theta.$$

$$= 96\pi - 256$$

Find the area inside the circle $r = 6\cos\theta$ and outside the cardioid $r = 2(1+\cos\theta)$



$r = 6\cos\theta = 2(1+\cos\theta)$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3.$

$$A = 2\int_0^{\pi/3} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta$$

$$= \int_0^{\pi/3} \left[(6\cos\theta)^2 - 4(1+\cos\theta)^2\right] d\theta = 4\pi.$$

Arc length :-

$$r = f(\theta) \quad , \quad \underline{a \le \theta \le b}$$

$x = f(\theta) \cos\theta$

$y = f(\theta) \sin\theta$.

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad , \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

So,

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

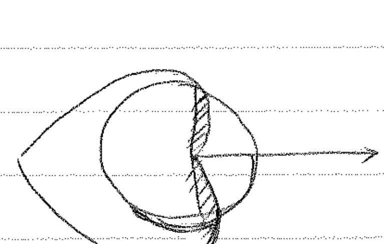
$r = e^{6\theta}$, $\quad 0 \le \theta \le 9\pi$

$$L = \int_0^{9\pi} \sqrt{e^{12\theta} + 36\, e^{12\theta}}\, d\theta$$

$$= \sqrt{37} \int_0^{9\pi} e^{6\theta}\, d\theta = \sqrt{37}/6 \left[e^{54\pi} - 1 \right]$$

Find the area shared by the circle $r=3$ and the cardioid $r = 3(1-\cos\theta)$

$$A = \tfrac{1}{2}\pi 3^2 + \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}\left(3(1-\cos\theta)\right)^2 d\theta .$$



$r = 2 - 2\sin\theta \cdot = 2(1-\sin\theta)$

$$2 \cdot \int_{-\pi/2}^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \underline{16}$$