118 **Power Series :-**

$\sum ar^n \equiv$ Geometric ~~ser~~ Series

$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \quad |r| < 1$ $\left[\sum_{n=1}^{\infty} \frac{3.2^n}{3(3^{2n})} \right]$

$a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots = \sum_{n=0}^{\infty} a_n r^n$ (Power Series)

$f(r) = \sum_{n=0}^{\infty} a_n r^n$, wherever the R.H.S. series converges.

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ (Polynomial ~~of~~ with infinite terms)

If $a_n = a$ for all n.

Power series $\equiv$ Geometric Series.

In general, $\sum_{n=0}^{\infty} a_n (x - c)^n$ is a power series centered at c, or power series ~~arong~~ around c.

$\sum_{n=0}^{\infty} a_n x^n$ is convergent for $x = 0$. Any other value ?

Exm:- $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$. $a_n = \frac{(x-2)^n}{n}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-2)^{n+1}}{(x-2)^n} \frac{n}{n+1} \right| \to |x-2| \quad \text{as } n \to \infty$$

So, by ratio test, the series is AC. if

$$|x-2| < 1$$
$$\Rightarrow \quad 1 < x < 3$$

So, the series converges for $1<x<3$ and diverges when $x<1$ or $x>3$

The ratio test fails for $|x-2| = 1$ i.e. $x = 3$ or 1.

If $x=3$, $\sum \frac{1}{n}$ diverges.

if $x=1$, $\sum (-1)^n \frac{1}{n}$ converges by Leibnitz test.

<u>Convergence</u> :- $\sum_{n=0}^{\infty} a_n (x-c)^n$. Three possibilities:

i) The series is convergent for only $x=c$.

ii) The series converges for all x.

iii) There is a +ve R such that, it converges if $|x-c|<R$ and diverges if $|x-c|>R$.

$\Rightarrow$ R is called the <u>radius of convergence</u>.

i) $R=0$.

ii) $R=\infty$

iii) For, $|x-c|=R$, we cannot say definitely!

<u>Test we use</u>: Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$.

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Exm :– $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1} \sqrt{n+1}}{\sqrt{n+2} \cdot (-3)^n x^n}\right| = \frac{|3x| \sqrt{n+1}}{\sqrt{n+2}} \to |3x| \text{ as } n \to \infty$$

By ratio test, the series converges if $|3x| < 1$

$\Rightarrow |x| < 1/3$ and

diverges if $|x| > 1/3$.

Radius of convergence $= \frac{1}{3}$.

Now, let, $|x| = 1/3$.

If $x = -\frac{1}{3}$, $\sum \frac{(-3x)^n}{\sqrt{n+1}} = \sum \frac{1}{\sqrt{n+1}}$ diverges.

If $x = \frac{1}{3}$, $\sum \frac{(-1)^n}{\sqrt{n+1}}$ converges by Leibnitz test

Exm :– $\sum \frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| \frac{n!}{(n+1)!} = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty$$

So, the power series converges for all x. $R = \infty$.

④

$$\sum_{n=2}^{\infty} (-1)^n \frac{(x-2)^{2n+1}}{n^2 \ln n}.$$

$$a_n = \frac{(-1)^n (x-2)^{2n+1}}{n^2 \ln(n)}, \quad c = 2.$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} (x-2)^{2n+3}}{(n+1)^2 \ln(n+1)} \cdot \frac{n^2 \ln(n)}{(-1)^n (x-2)^{2n+1}}\right|$$

$$= (x-2)^2 \frac{n^2}{(n+1)^2} \frac{\ln(n)}{\ln(n+1)} \to (x-2)^2 \text{ as } n \to \infty$$

The power series converges if $(x-2)^2 < 1 \Rightarrow |x-2| < 1$.

So, $R = 1$. and domain of convergence $-1 \leq x \leq 3$,

for $|x-2| = 1$ the series becomes $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 \ln n}$. AC so convergent.

Exm: $$\sum_{n=0}^{\infty} \frac{(2x+4)^{3n}}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2^3}{3}\right)^n (x+2)^{3n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2^3)^{n+1}}{3^{n+1}} \frac{(x+2)^{3n+3}}{2^{3n}} \cdot \frac{3^n}{(x+2)^{3n}}\right| = \left|\frac{8}{3} (x+2)^3\right| \to \frac{8}{3} |x+2|^3.$$

So, PS. converges if $\frac{8}{3} |x+2|^3 < 1$

$$\Rightarrow |x+2| < \frac{\sqrt[3]{3}}{2} = R$$

\# Suppose $\sum a_n x^n$ converses at $x=-3$ and diverses at $x=8$.
What can you say?

R is the Radius of convergence.

$|x|<R$. then, $+3 \leqslant R < 8$.

So, $\sum a_n 4^n$ is non-conclusive.

but, $\sum a_n 2^n$ is conversent.

\# $\sum_{n=0}^{\infty} n!\,(x-6)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x-6)^{n+1}}{n!\,(x-6)^n}\right| = (n+1)\,|x-6| \to \infty \quad \text{as } n \to \infty$$

for $x \neq 6$.

So, divergent except $x=6$.

\# $$\sum_{n=1}^{\infty} \frac{(2x-3)^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{2^{2n+1}}{n^{3/2}} \left(x-\tfrac{3}{2}\right)^{2n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = 2^2 \left(x-\tfrac{3}{2}\right)^2 \cdot \frac{n^{3/2}}{(n+1)^{3/2}} \to 4\left(x-\tfrac{3}{2}\right)^2$$

Convergent if $4\left(x-\tfrac{3}{2}\right)^2 < 1 \Rightarrow \left|x-\tfrac{3}{2}\right| < \tfrac{1}{2}$

$R = \tfrac{1}{2}$, $1 < x < 2$.

$$1 + x + x^2 + \cdots = \frac{1}{1-x}, \quad |x| < 1.$$

$$\frac{x^2}{4+x^2} = \frac{x^2}{4} \cdot \frac{1}{1+x^2/4} = \frac{x^2}{4}\left(1 + \frac{x^2}{4}\right)^{-1} \qquad (1-r)^{-1}, \; r = -\frac{x^2}{4}.$$

$$= \frac{x^2}{4} \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n \qquad \text{for } \left|-\frac{x^2}{4}\right| < 1.$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^2}{4}\right)^{n+1} = \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{2n+2}}}$$

$\underline{\underline{R}}$ $\quad |x^2| < 4.$ i.e. $\underline{|x| < 2}$

$$\underline{\underline{R = 2}}$$

Differentiation & Integration :-

Let, $\sum_{n=0}^{\infty} a_n (x-c)^n$ has RC $R > 0$, then,

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

f is differentiable on $(c-R, c+R)$ and

i) $\quad f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$ (Term by term diff.)

ii) $\quad \int f(x)\,dx = K + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$

Radius of convergence of (i) & (ii) is R.

\# $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n \quad \cdots (*)$$

for $|x|<1$.

\# $$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \quad \text{for } |x|<1$$

$$= \sum_{n=0}^{\infty} (-x)^n$$

Integrate, $$\int \frac{dx}{1+x} = K + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\Rightarrow \ln|1+x| = K + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Put $x=0 \Rightarrow 0 = K$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x|<1$$

\# $$\frac{6}{(1-x)^3} = 6(1-x)^{-3} = 3.2(1-x)^{-2} = 3\sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$= 6 + 18x + 36x^2 + \cdots$$

Valid for $|x|<1$.

$f(x) = \tan^{-1}(2x)$. as power series.

$f'(x) = \frac{2}{1+4x^2} = 2(1+4x^2)^{-1}$

$= 2 - 8x^2 + 32x^4 - \cdots$ for $|x| < \frac{1}{2}$

So, $2\int \frac{dx}{1+4x^2} = \int (2 - 8x^2 + 32x^4 - \cdots)\, dx$

$\Rightarrow \tan^{-1}(2x) = k + 2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \cdots$ for $|x| < 1/2$

$k = 0$ at $x = 0$.

So,

$\tan^{-1}(2x) = 2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \cdots \quad |x| < 1/2$