* $\displaystyle\int \frac{\log_7 x}{x}\,dx = \int \frac{\ln(x)}{\ln 7 \cdot x}\,dx$ let, $\ln x = u$

$\frac{1}{x}dx = du$.

$= \displaystyle\frac{1}{\ln(7)}\int u\,du$

$= \frac{1}{2}u^2 \frac{1}{\ln 7} = \frac{(\ln x)^2}{2\ln(7)}$.

Q.H.W: Find $\frac{dy}{dx}$, $y = x^x$

Exponential Growth & Decay:-

Consider a fn $x = x(t)$ satisfying

I) the rate of change of x is proportional to x

II) the initial value of x is $x(0) = c$.

Find x.

$$\frac{dx}{dt} = kx$$

$\Rightarrow x(t) = ce^{kt}$ is the only soln

General Method: (Seperable equation)

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}.$$

$\Rightarrow f(y)\,dy = g(x)\,dx$.

$\Rightarrow \int f(y)\,dy = \int g(x)\,dx + c$

Exm: $f'(x) = \dfrac{\sin x}{f(x)}$

Let, $y = f(x)$.

$$\Rightarrow y' = \frac{\sin x}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{y} \Rightarrow \int y\,dy = \int \sin x\,dx + c$$

$$\Rightarrow \frac{y^2}{2} = -\cos x + c.$$

$$\Rightarrow y = \sqrt{2c - 2\cos x} \quad \Rightarrow f(x) = \sqrt{2c - 2\cos x}$$

Newton's law of cooling:

Temperature $T(t)$ of a hot body cooling down to environmental temperature E. The rate of cooling is proportional to the diff. between the body temp & E.

$$\frac{dT}{dt} = -K(T-E)$$

$$\Rightarrow \int \frac{dT}{T-E} = -K\int dt + c$$

$$\Rightarrow \ln|T-E| = -Kt + c$$

$$\therefore T = E \pm e^{-Kt} \cdot e^{c}$$

$$T = E + \mu e^{-Kt}$$

exponential decay.

\# In a room ~~of~~ at 20°c, a cup of boiling tea (100°c) cools to 80°c in 1 min. How long until it is sippable at 50°c?

$$\underline{T(t) = E + \mu e^{-kt}}$$ by Newton's law.

$$\Rightarrow T(t) = \cancel{80}\ 20 + \mu e^{-kt}$$

$T(0) = 100 \Rightarrow 20 + \mu = 100 \Rightarrow \mu = 80$

$T(1) = 80 \Rightarrow 20 + 80e^{-k} = 80$

$\Rightarrow e^{-k} = 60/80 = 3/4$.

$\therefore T(t) = 20 + 80\left(3/4\right)^t$.

$\therefore T(t) = 50 \Rightarrow 50 = 20 + 80 \cdot \left(3/4\right)^t$

$\Rightarrow 8\cancel{0}\left(3/4\right)^t = 3\cancel{0}$

$\Rightarrow \left(3/4\right)^t = 3/8$

$$\Rightarrow t = \underline{\underline{\frac{\ln(3/8)}{\ln(3/4)}}}$$

11 $\quad \dfrac{dy}{dx} = \dfrac{x}{y}, \ y(0) = -3$

$\Rightarrow y\,dy = x\,dx$

$\Rightarrow \dfrac{y^2}{2} = \dfrac{x^2}{2} + K$

$\Rightarrow y^2 = x^2 + c$

$\Rightarrow (-3)^2 = c \Rightarrow c = 9$

$\therefore y^2 = x^2 + 9$

$\therefore y = \pm\sqrt{x^2+9}$ By initial condn. $\underline{y(x) = -\sqrt{x^2+9}}$

(4)

Ex $$\frac{dy}{dx} = e^{-y}(2x-4)$$

$$\Rightarrow e^{y}\,dy = (2x-4)\,dx \quad \Rightarrow \underline{e^{y} = x^{2} - 4x + c}$$

Let, $y(5) = 0$

$$\Rightarrow 1 = 25 - 20 + c$$

$$\Rightarrow c = -4$$

$$\underline{e^{y} = x^{2} - 4x - 4}$$

$$\therefore y = \ln|x^{2} - 4x - 4|$$

20 $$f'(x) = f(x)\big(1 - f(x)\big), \quad f(0) = \tfrac{1}{2}$$

Let, $y = f(x)$

$$\Rightarrow y' = y(1-y)$$

$$\Rightarrow \frac{dy}{y(1-y)} = dx$$

$$\Rightarrow \int \left[\frac{1}{y} + \frac{1}{1-y}\right] dy = \int dx + c$$

$$\Rightarrow \ln|y| + \ln|1-y| = x + c$$

$$\Rightarrow y(1-y) = e^{x+c} = Ke^{x}$$

$$\therefore \tfrac{1}{2}\left(1 - \tfrac{1}{2}\right) = K \Rightarrow K = \tfrac{1}{4}$$

$$\therefore y(1-y) = \tfrac{1}{4}e^{x}$$

$$y^{2} - y + \tfrac{1}{4}e^{x} = 0 \Rightarrow f(x) = \frac{1 \pm \sqrt{1 - e^{x}}}{2}$$

$$f(0) = \tfrac{1}{2} \Rightarrow f(x) = \frac{1 \pm \sqrt{1 - e^{x}}}{2}$$

Ex.

$$y' = e^{4x-3y}, \quad y(0) = 0$$

$$\Rightarrow \quad e^{3y}\,dy = e^{4x}\,dx.$$

$$\Rightarrow \quad \frac{1}{3}e^{3y} = \frac{1}{4}e^{4x} + c.$$

$$\Rightarrow \quad e^{3y} = \frac{3}{4}e^{4x} + K. \quad \cancel{\Rightarrow 3y = \ln(3/4) + 4x}$$

$$1 = \frac{3}{4} + K \quad \Rightarrow K = 1/4.$$

$$\therefore 4e^{3y} = 3e^{4x} + 1 \quad \therefore \; y = \frac{1}{3}\ln\left(\frac{3}{4}e^{4x} + \frac{1}{4}\right)$$

Ex A scientist receiving a new sample of plutonium knows that the sample will not be of use to him after 85% of the material is disintegrated. Given

i) Half-life is 139 days

How ~~may~~ many days will be of use?

Soln:
$$\frac{dm}{dt} = km$$

$$\Rightarrow \frac{dm}{m} = k\,dt \quad \Rightarrow \ln|m| = kt + c$$

$$\therefore \; m = \mu e^{kt}$$

$$m(t) = m(0)\,e^{kt}.$$

$$\frac{1}{2} = e^{k \cdot 139}$$

$$\Rightarrow \; 139\,k = -\ln 2.$$

$$\therefore \; k = -\ln 2/139$$

$$\frac{15}{100}\cancel{m(0)} = \cancel{m(0)}\,e^{-\frac{\ln 2 \cdot t}{139}}$$

$$\therefore -\frac{\ln 2}{139}\,t = \ln\left(\frac{15}{100}\right)$$