11.1 Sequences

Sequence is a list of numbers in a particular order.

$\{a_n\}_n$: 1, 2, 3, 4, 5, . . . , n, . . .

↑ 1st term ↑ nth term.

$\{b_n\}_n$ 1, 3, 2, 5, 4, .

Can be given as a function $f: \mathbb{N} \to \mathbb{R}$.

$f(n) = n$. , $n = 1, 2, \ldots$

$f(n) = 2n$, $n = 1, 2, \ldots$

$f(n) = n^2$.

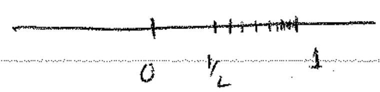
⋮

Exm:- Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, . . .

$f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$

Exm: $f(n) = \dfrac{n}{n+1}$ $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\}$



$\lim_{n\to\infty} \dfrac{n}{n+1} = 1$. i.e. $\lim_{n\to\infty} a_n = 1$

Defn:- A sequence $\{a_n\}_n$ has the limit L, and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for any $\epsilon > 0$ (small), there is a large N_0 such that

$$|a_n - L| < \epsilon \quad \text{for } n \geq N_0$$

So, $|a_n - 1| = \left| \frac{n}{n+1} - 1 \right|$

$$= \frac{1}{n+1} < \epsilon$$

for $n+1 > \frac{1}{\epsilon}$

$\Rightarrow n \geq \frac{1}{\epsilon} - 1$ $\quad N_0 = \frac{1}{0.01} - 1 = 100 - 1$

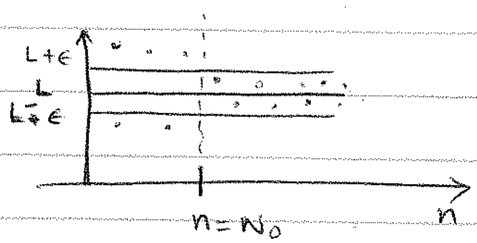
$= 99.$

with $\epsilon = 0.01$

We say $\{a_n\}$ to be convergent sequence. (or divergent seq$\underline{n}$)

$\Rightarrow \quad |a_n - L| < \epsilon \quad$ for $n \geq N_0$

$\Rightarrow \quad L - \epsilon < a_n < L + \epsilon \quad$ for $n \geq N_0$.



In particular, if $\lim_{x \to \infty} f(x) = L$, then, $\lim_{n \to \infty} f(n) = L$.

Def$\underline{n}$:- $\lim_{n \to \infty} a_n = \infty$ if for every large number M there is N_0 such that.

$$a_n > M \quad \text{for } n \geq N_0$$

Properties :-

i) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

ii) $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.

iii) $\lim_{n\to\infty} c\,a_n = c \lim_{n\to\infty} a_n$, c = const.

iv) $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

v) $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \Big/ \lim_{n\to\infty} b_n$ if $\lim_{n\to\infty} b_n \neq 0$.

6 **Squeeze theorem (Sandwich theorem).:**

If $a_n \le b_n \le c_n$. for $n \ge N_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then, $\lim_{n\to\infty} b_n = L$.

$\lim_{n\to\infty} \frac{\ln n}{n}$. $\left(\frac{\infty}{\infty}\right)$. l' Hospital

Consider $f(x) = \frac{\ln x}{x}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$a_n = (-1)^n$.

$\{-1, 1, -1, 1 \ . \ . \ . \}$.

Not convergent.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Exm: $a_n = \frac{(-1)^n}{n}$.

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$.

So, $\lim_{n\to\infty} a_n = 0$.

Counter-Exm: $a_n = (-1)^n$. $\lim_{n\to\infty} |a_n| = 1$ but $\{a_n\}_n$ is divergent.

Thm:- If $\lim_{n\to\infty} a_n = L$ and f is a continuous fn at L, then, $\lim_{n\to\infty} f(a_n) = f(L)$

$$\lim_{n\to\infty} \sin(\pi/n^2) = \cancel{\lim_{n\to\infty}} \sin\left(\lim_{n\to\infty} \pi/n^2\right) = \sin(0) = 0.$$

Exm:- $a_n = \frac{n!}{n^n} = \frac{1.2.\cdots n}{n.n.n.\cdots n} = \frac{1}{n}\left(\frac{2.3.\cdots n}{n.n.\cdots n}\right) \le \frac{1}{n}$.

So, $0 < a_n \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

So, $\lim_{n\to\infty} a_n = 0$ by Squeeze theorem.

Exm:- $a_n = r^n$. $\lim_{n\to\infty} a_n = \begin{cases} 0, & 0<r<1 \\ \infty, & r>1 \end{cases}$

Defn: $\{a_n\}_n$ is increasing if $a_n \leq a_{n+1}$ for $n \geq 1$.

$\{a_n\}_n$ is decreasing if $a_n \geq a_{n+1}$ for $n \geq 1$.

$\{a_n\}_n$ is monotone if it is either increasing or decreasing

Exm: $a_n = \frac{1}{n}$. $a_n = \frac{1}{n} \geq \frac{1}{n+1} = a_{n+1}$, $n \geq 1$.

So, decreasing.!

Defn:- A sequence $\{a_n\}_n$ is bounded above if

$$a_n \leq M \quad \text{for all } n.$$

& bounded below if

$$a_n \geq m \quad \text{for all } n.$$

$\{a_n\}_n$ is bounded if $m \leq a_n \leq M$ for all n.

\# Monotonic Sequence theorem:- Every bounded, monotonic sequence is convergent.

Proof:- Let, $\{a_n\}_n$ is increasing.

$$\text{So, } a_n \leq a_{n+1} \quad \text{for } n \geq 1.$$

Also, it is bounded, so,

$$m \leq a_n \leq M \quad \text{for } n \geq 1.$$

Conside $S = \{a_n : n \in \mathbb{N}\}$. S is bounded and by completeness axiom, it has a least upper bound L.

Now, given $\epsilon > 0$, there is a point a_N such that,

$$L - \epsilon < a_N \leq L$$

i.e. $L - \epsilon < a_N \leq a_{N+1} \cdots \leq L$

i.e. $L-\epsilon < a_n \leq L < L+\epsilon \quad , \; n \geq N$

$$\Rightarrow \quad |a_n - L| < \epsilon \quad \text{for } n \geq N$$

i.e. $\lim\limits_{n\to\infty} a_n = L$

\# $-1, \frac{1}{2^4}, -\frac{1}{3^4}, \frac{1}{4^4}, -\frac{1}{5^4} \ldots$

$$a_n = (-1)^n \frac{1}{n^4}$$

\# $a_n = \frac{5-5n}{4+2n}$ $\quad \lim\limits_{n\to\infty} \frac{5-5n}{4+2n} = \lim\limits_{n\to\infty} \frac{5/n - 5}{4/n + 2} = -5/2.$

\# $a_n = \frac{\ln(1-9/n)}{\sin(\frac{1}{n})}$

$$\lim_{x\to\infty} \frac{\ln(1-9/x)}{\sin(1/x)} \left(\frac{0}{0}\right) = -\lim_{x\to\infty} \frac{\frac{1}{1-9/x} \cdot \frac{9}{x^2}}{\cos(\frac{1}{x}) \cdot 1/x^2} = -9.$$

\# $a_n = \frac{3\cos(n^6)}{\sqrt{n}}$ $\quad 0 \leq a_n \leq \frac{1}{\sqrt{n}}.$

Sandwich thm $\Rightarrow \lim\limits_{n\to\infty} a_n = 0.$

\# $\lim\limits_{n\to\infty} \frac{8+5/n}{(2n)^{1/n}} = \frac{\lim\limits_{n\to\infty} 8+5/n}{\lim\limits_{n\to\infty} (2n)^{1/n}}$

$$y = \lim_{x\to\infty} (2x)^{1/x} \Rightarrow \ln y = \lim_{x\to\infty} \frac{1}{x}\ln(2x)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2x}\cdot 2}{1} = 0$$

$\therefore y = 1$

So, $\lim\limits_{n\to\infty} a_n = \frac{8}{1} = 8.$

43 $a_n = \frac{\cos^2 n}{2^n}$

$0 \le a_n \le \frac{1}{2^n} \to 0$ by Sandwich.

45. $a_n = n \sin\left(\frac{1}{n}\right)$ $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$

$= 1.$

\# $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$ $n! = 1.2.3.\ldots.n$ $0! = 1$

11.2. Series

$\{a_n\}_n$: Sequence.

$a_1 + a_2 + a_3 + a_4 + \cdots$: Series.

$\sum a_n$ or $\sum_{n=1}^{\infty} a_n$

\# $1+2+3+\cdots+n+\cdots = \sum a_n$

$S_n = 1+2+\cdots+n$

$= n(n+1)/2$: Sequence of partial sums.

$\{S_n\}_n$ converges if and only if the series converges.

~~Green~~ Exm :- $S_n = \frac{2n}{3n+5}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{3+5/n} = 2/3.$ So, the series $\sum a_n$ converges.