43 $a_n = \frac{\cos^2 n}{2^n}$.

$0 \le a_n \le \frac{1}{2^n} \to 0$ by Sandwich.

45. $a_n = n \sin\left(\frac{1}{n}\right)$ $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$

$= 1$.

\# $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. $n! = 1.2.3.\ldots.n$ $0! = 1$

11.2. Series

$\{a_n\}_n$: Sequence.

$a_1 + a_2 + a_3 + a_4 + \cdots$: Series.

$\sum a_n$ or $\sum_{n=1}^{\infty} a_n$

\# $1+2+3+\cdots+n+\cdots = \sum a_n$

$S_n = 1 + 2 + \cdots + n$

$= n(n+1)/2$: Sequence of partial sums.

$\{S_n\}_n$ converges if and only if the series converges.

~~Theorem~~ Exm :- $S_n = \frac{2n}{3n+5}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{3+5/n} = 2/3$. So, the series $\sum a_n$ converges.

Geometric series:

$$a_n = ar^{n-1}, \quad n = 1, 2, \ldots$$

$$a + ar + ar^2 + \cdots$$

$r = 1 \Rightarrow \quad \sum a_n = a + a + \cdots + a \ldots$

$$S_n = na \to \pm\infty \text{ as } n \to \infty.$$

$r \neq 1$, $\quad S_n = a + ar + \cdots + ar^{n-1}$

$$\therefore rS_n = ar + ar^2 + \cdots + ar^n$$

$$\Rightarrow S_n(1-r) = a - ar^n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Now, if $-1 < r < 1$ $(|r| < 1)$, $r^n \to 0$ as $n \to \infty$

So,

$$\lim_{n \to \infty} S_n = \frac{a}{1-r}, \quad |r| < 1$$

For, $r \leq -1$ or $r > 1$, $\{r^n\}$ diverges, so, $\{S_n\}_n$ diverges

\# $\sum 2^{2n} 3^{1-n} = \sum 4^n \cdot \frac{1}{3^{n-1}} = \sum 3\left(\frac{4}{3}\right)^n$

$r = 4/3 > 1$, divergent.

See Exm. 2·3|7

③

Harmonic Series:

$$\sum \frac{1}{n}.$$

$$S_2 = 1+\frac{1}{2}$$

$$S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+\frac{2}{2}.$$

$$S_8 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) > 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{3}{2}.$$

$$S_{16} > 1+\frac{4}{2}$$

So, $S_{2^n} > 1+\frac{n}{2}$.

$\therefore$ $S_{2^n} \to \infty$ as $n \to \infty$, so $\{S_n\}$ is divergent.

So, harmonic series diverges.

Thm: If $\sum a_n$ converges, $a_n \to 0$ as $n \to \infty$. $\left(\lim_{n\to\infty} a_n = 0.\right)$

Pf:- $S_n = a_1 + a_2 + \cdots + a_n$.

So, $S_n - S_{n-1} = a_n$. Since, $\sum a_n$ converges, $\{S_n\}_n$ is convergent.

So, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$

Converse Not True: $\sum \frac{1}{n}$. $\frac{1}{n} \to 0$ as $n \to \infty$.

(4)

Show, $0.9999\ldots = 1$

$$0.9 + 0.09 + 0009 + \cdots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right)$$

$$= \frac{9}{10} \cdot \frac{1}{1 - 1/10} = 1.$$

$$\sum a_n \text{ conv.} \Rightarrow a_n \to 0$$

$$\Rightarrow a_n \not\to 0 \Rightarrow \sum a_n \text{ diverges.}$$

<u>Telescoping Series</u> : $\sum \frac{6}{4n^2-1}$

$$\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$\Rightarrow A = \frac{6}{1+1} = 3$$

$$\Rightarrow B = \frac{6}{-1-1} = -3$$

$$S_n = \sum_{n=1}^{n} \frac{6}{4n^2-1} = \sum_{n=1}^{n}\left(\frac{3}{2n-1} - \frac{3}{2n+1}\right) = \left(3 - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{5}}\right)$$

$$+ \left(\cancel{\frac{3}{5}} - \cancel{\frac{3}{7}}\right) + \cdots + \left(\cancel{\frac{3}{2n-1}} - \frac{3}{2n+1}\right)$$

$$= 3 - \frac{3}{2n+1} \to 3 \text{ as } n \to \infty.$$

⑤

$0.444\ldots = 0.\overline{4} = \dfrac{4-0}{10-1} = \dfrac{4}{9}$. (Easy rule)

$1.\overline{554} = \dfrac{1554-1}{1000-1} = \dfrac{1553}{999}$.

43. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$= \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}\right)$
$\ldots + \left(\tfrac{1}{n-1} - \tfrac{1}{n+1}\right)$

41. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$

$\displaystyle\sum \frac{1}{e^n} + \sum \frac{1}{n(n+1)}$.

$\downarrow \qquad \searrow$

$\dfrac{1}{e}\dfrac{1}{1-1/e} \qquad \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)$

$= 1 - \dfrac{1}{n+1} \to 1$ as $n \to \infty$

$\text{Sum} = \dfrac{1}{e-1} + 1 = \dfrac{e}{e-1}$