MTH 133-60 Lecture Notes

11.7 Strategy of testing series

I. $\sum \frac{1}{n^p}$ C : $p>1$ and $p\le 1$: D

II. $\sum ar^{n-1}$ C : $|r|<1$ and D : $|r|\geq 1$.

III. $\frac{P(n)}{Q(n)} \sim \frac{1}{n^p}$. $\sum \frac{n^5+10n+4}{n^{10}-2n-1} \sim \sum \frac{n^5}{n^{10}} = \sum \frac{1}{n^5}$

IV. If $\lim_{n\to\infty} a_n \neq 0$, D.

V. $\sum (-1)^n a_n$: Try Leibnitz alternating ~~ts~~ Series test.

VI. $\left|\frac{a_{n+1}}{a_n}\right|$ or $|a_n|^{1/n}$ (Ratio or root test)

$\left|\frac{a_{n+1}}{a_n}\right|$ ← $n!$

$|a_n|^{1/n}$ ↳ $a_n = (b_n)^n$.

VII. $a_n = f(n)$: $\int_1^\infty f(x)$ Integral test. (positive series)

ExM :- $\sum \frac{4^n(n!)^2}{(2n)!}$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}((n+1)!)^2 \; (2n)!}{(2n+2)! \; 4^n (n!)^2}$$

$$= \frac{4(n+1)^2}{(2n+2)(2n+1)} = \frac{2n+2}{2n+1} > 1$$

$\Rightarrow a_{n+1} > a_n$ for all n.

And, $a_1 = \frac{4}{2} = 2$ So, $a_n \not\to 0$. as $n\to\infty$

So, $\sum a_n$ is divergent.

②

\# $\sum \frac{n^2}{2^n}$

$$a_n = \frac{n^2}{2^n}. \qquad b_n = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^2 = \infty$$, doesn't work.

$$c_n = \left(\frac{3}{4}\right)^n. \qquad \lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} n^2 \left(\frac{2}{3}\right)^n = 0$$

So, by comparison test, $\sum a_n$ is convergent.

$$\lim_{x\to\infty} \frac{x^2}{r^x} = \lim_{x\to\infty} \frac{2x}{r^x \ln r} = \lim_{x\to\infty} \frac{2}{r^x (\ln r)^2} = 0 \quad (r>1)$$

\# $$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2^{n+1} n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

~~Not convergent as~~ Not, AC. as

$$a_n = \frac{1}{2n} \qquad b_n = \frac{1}{n} \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}.$$

(comparison test)

\# $\sum \frac{1}{n\sqrt{2n+1}}$

$$\frac{1}{n\sqrt{2n+1}} \le \frac{1}{n\sqrt{2n}} = \frac{1}{\sqrt{2}\, n^{3/2}}$$

$\sum \frac{1}{\sqrt{2}\, n^{3/2}}$ converges as it is a p-series with $p = 3/2 > 1$

So, by comparison test, $\sum \frac{1}{n\sqrt{2n+1}}$ converges.

$$\# \quad \sum \frac{2n(n+1)}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2(n+1)(n+2)\,n!}{(n+1)!\;2n(n+1)} = \frac{n+2}{n(n+1)} \to 0 \text{ as } n\to\infty.$$

So, by ratio test, the series converges.

$$\# \quad \text{i)} \sum \frac{2n}{n+3} \qquad \text{ii)} \sum \frac{-8}{(-3)^n}, \qquad \text{iii)} \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

i) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{2n}{n+3} = \lim\limits_{n\to\infty} \frac{2}{1+3/n} = 2 \neq 0$. diverges.

ii) $-8 \sum \frac{1}{(-3)^n} = \sum ar^{n-1} = +8 \cdot \frac{+1/3}{1+1/3} = 2$.

iii) $\sum\limits_{n=0}^{\infty} \frac{1}{2^n} = \cancel{6}\, \frac{1}{1-1/2} = 2$.

$$\# \quad 1 + 2^x + 3^x + \cdots + n^x \quad \text{as } n\to\infty$$

$= \sum\limits_{n=1}^{\infty} n^x \approx \sum \frac{1}{n^{-x}}$. converges for $-x > 1$

i.e. $x < -1$

$$\# \quad \sum \frac{n!}{e^n} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{e} \to \infty.$$

$\sum \frac{n+2}{n^2-n}$

$b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n}{n^2-n} = 1.$

So, by limit comparison test, $\sum a_n$ diverges.

$\sum_{n=0}^{\infty} \frac{3-2^{n+1}}{7^n} = \sum_{n=0} \frac{3}{7^n} - \sum_{n=0} 2\left(\frac{2}{7}\right)^n$

$= \frac{3}{1-\cancel{2}1/7} - \frac{2}{1-2/7} = \frac{21}{6} - \frac{14}{5}$

$\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n}$

Integral test : $\int_{x=1}^{\infty} \frac{\ln(x^2)}{x} dx$ let, $\ln(x^2) = u$

$\frac{1}{x^2} 2x\,dx = du$

$= \lim_{N\to\infty} \int_0^{\ln(N^2)} \frac{u}{2} du = \lim_{N\to\infty} \frac{1}{4} (\ln(N^2))^2 = \infty$

So, divergent.

$\sum \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ $a_n = \frac{1}{\sqrt{n}(\sqrt{n}+1)}$, $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = 1.$

So, $\sum a_n$ diverges ~~and~~ as $\sum \frac{1}{n}$ diverges (Limit comparison test.)

$\sum \sin(\frac{1}{n})$ $\qquad$ $\sum \cos(\frac{1}{n})$.

$\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$. , non-conclusive.

$\lim_{n\to\infty} \cos(\frac{1}{n}) = 1 \neq 0$. divergent.

$\sum \sin(n)$. Can Integral test be applied?

$f(x) = \sin(x)$. is not decreasing or +ve.

Not ~~applied!~~ applicable!

$\sum \frac{1}{n+2}$ $\qquad$ $\sum \frac{n^2}{2n^3-1}$

$b_n = \frac{1}{n}$. Apply limit comparison test.

Both are divergent.

$\sum_{k=1}^{\infty} (-1)^k \frac{1}{2^k-3}$ $\qquad$ $a_n = \frac{1}{2^n-3}$.

Consider. $\sum_{n=2}^{\infty} (-1)^n a_n$ $\qquad$ $b_n = \frac{1}{2^n}$.

Geometric series

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. $\qquad$ $\sum b_n$ is convergent, $\sum ar^{n-1}$ $r = \frac{1}{2}$.

So, by limit comparison test, $\sum_{n=2}^{\infty} |a_n|$ is convergent, so does

~~$\sum a_n$~~ $\sum (-1)^n a_n$

\# $\displaystyle\sum_{n=0}^{\infty} \frac{2-3^{n+1}}{6^n} = \sum_{n=0}^{\infty} \frac{2}{6^n} - \sum_{n=0}^{\infty} 3\left(\frac{3}{6}\right)^n$

$= \dfrac{2}{1-1/6} - \dfrac{3}{1-1/2} = \dfrac{12}{5} - 6 = -\dfrac{18}{5}.$

\# $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+n}{n^4+\sqrt{n}}$ $\qquad b_n = \dfrac{3n^2}{n^4} = \dfrac{3}{n^2}.$

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$

$\sum b_n$ is a p-series $p>2$. So, $\sum a_n$ is convergent.