11.10 Taylor and Maclaurin Series :-

For a given function $f(x)$, lets write

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \qquad |x-c| < R$$

To find a_n's.

$$f(c) = a_0$$
$$f'(c) = a_1$$
$$f''(c) = 2a_2$$
$$\vdots$$
$$f^{(n)}(c) = n!\, a_n$$

Result: If $f(x)$ has a power series representation at c

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \qquad |x-a| < R$$

then, $a_n = \dfrac{f^{(n)}(c)}{n!}$.

i.e. $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

This is Taylor series of f at c.

If $c = 0$, $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This is called Maclaurin Series.

$f(x) = e^x$.

$$f^{(n)}(x) = e^x \quad \text{for all } n.$$

$$\therefore f^{(n)}(0) = 1.$$

$$\text{So, } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Radius of convergence:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{n+1} \to 0 \text{ as } n \to \infty$$

So, by ratio test, the series converges for all x. $R = \infty$.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{n!} (x-c)^i$$

nth degree Taylor polynomial of f.

$$f(x) = \lim_{n \to \infty} T_n(x). \quad \text{if } f \text{ has a power series represent}$$

$$R_n(x) = f(x) - T_n(x) \quad \text{so that, } f(x) = T_n(x) + R_n(x)$$

$R_n(x)$ is called the remainder of the Taylor series.

If $\lim_{n \to \infty} R_n(x) = 0$ then,

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} \left[f(x) - R_n(x) \right] = f(x)$$

<u>Taylor's Inequality</u> :- If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad , \quad \text{for } |x-a| \leq d.$$

$f(x) = \sin(x)$.

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{(4)}(x) = +\sin x.$$

Maclaurin Series $= f(0) + f'(0).x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}}{4!}x^4 + \dots$

$$= x - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$f^{(n+1)}(x) = \pm \sin x \text{ or } \pm \cos x$$

$$\therefore \left|f^{(n+1)}(x)\right| \leq 1$$

So $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

So, $R_n(x) \to 0$ as $n \to \infty$.

Therefore, $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

\# $g(x) = \cos(x)$

$$\cos(x) = \frac{d}{dx}(\sin x)$$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ for all } x.$$

\# Find Taylor series of $\sin(x)$ centered at $\pi/4$.

$f(x) = \sin x$	$f(\pi/4) = \frac{1}{\sqrt{2}}$
$f'(x) = \cos x$	$f'(\pi/4) = 1/\sqrt{2}$
$f''(x) = -\sin x$	$f''(\pi/4) = -1/\sqrt{2}$
$f'''(x) = -\cos x$	$f'''(\pi/4) = -1/\sqrt{2}$
$f^{(4)}(x) = \sin x$	$f^{(4)}(\pi/4) = \frac{1}{\sqrt{2}}$

$$\text{T. Series} = f(\pi/4) + (x-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4)$$
$$+ \frac{(x-\pi/4)^3}{3!} f'''(\pi/4) + \frac{(x-\pi/4)^4}{4!} f^{(4)}(\pi/4)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x-\pi/4) - \frac{1}{\sqrt{2}}\frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!}\frac{1}{\sqrt{2}}$$
$$+ \frac{1}{\sqrt{2}}\frac{(x-\pi/4)^4}{4!} + \cdots$$

(5)

$$f(x) = (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n. \quad \text{(Sigma notation)}$$

$$= 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots \quad \text{for } |x|<1$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)!}{n!} = \frac{k!}{n!\,(k-n)!}$$

$$\frac{1}{\sqrt{a-x}} = (a-x)^{-1/2} = \frac{1}{\sqrt{a}}\left(1-\frac{x}{a}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{a}} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(-\frac{x}{a}\right)^n$$

$$\text{for } \left|-\frac{x}{a}\right| < 1$$

$$\text{i.e. } \underline{|x| < a}.$$

$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{for } |x|<1$$

$$(1-x)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} (-x)^n$$

$$= 1 + (-k)(-x) + \frac{(-k)(-k-1)}{2!}(-x)^2 + \frac{(-k)(-k-1)(-k-2)}{3!}(-x)^3 + \dots$$

$$= 1 + kx + \frac{k(k+1)}{2!} x^2 + \frac{k(k+1)(k+2)}{3!} x^3 + \dots$$

$f(x) = \frac{1}{x}$. $\quad c = 2$.

$$f'(x) = -1/x^2, \quad f^{(2)}(x) = 2/x^3, \quad f^{(3)}(x) = -6/x^4 \ \ldots \ f^{(n)}(x) = (-1)^n \frac{n!}{x^{(n+1)}}$$

So, $$\frac{1}{x} = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \cdots$$

Geometric series with $r = -\frac{(x-2)}{2}$. So, convergent if

$$\left| -\frac{x-2}{2} \right| < 1 \implies |x-2| < 2.$$

$$\therefore 0 < x < 4.$$

and the Sum $= \frac{a}{1-r} = \frac{1/2}{1+\frac{x-2}{2}} = \frac{1}{x}$.

<u>Taylor Formula</u> :-

$$\cancel{R_n(x)}$$

$$f(x) = T_n(x) + \frac{f^{(n+1)}(k)}{(n+1)!}(x-c)^{n+1}, \quad c < k < x.$$

$f(x) = \frac{10}{(1-6x)^3}$.

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{(Sigma notation)} \quad |x| < 1$$

$$g'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

$$g''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 \cdots = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$g''(6x) = \frac{2}{(1-6x)^3} = 2 + 6\cdot(6x) + 12\cdot(6x)^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)(6x)^{n-2}$$

$$5g''(6x) = f(x) = 10 + 30\cdot(6x) + 60(6x)^2 + \cdots = \sum_{n=2}^{\infty} 5n(n-1)(6x)^{n-2}$$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$x = i\theta$.

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta.$$

$$\therefore\ e^{i\theta} = \cos\theta + i\sin\theta$$

$\theta = \pi \Rightarrow \quad e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$\Rightarrow \underline{\underline{e^{i\pi} + 1 = 0}} \qquad (\text{Euler's identity})$$

11.11

Applications :

Write $f(x) = x^3 - 2x + 4$ in terms of $(x-2)$

$f'(x) = 3x^2 - 2 \qquad f'(2) = 10$

$f''(x) = 6x \qquad f''(2) = 12$

$f'''(x) = 6. \qquad f'''(2) = 6.$

So, $f(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3.$

$$\frac{x}{x^2+4} = \frac{x}{4}\left(1+\left(\frac{x}{2}\right)^2\right)^{-1} = \frac{x}{4}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{2}\right)^2\right]^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{4^{n+1}}$$

\# $f(x) = 10x\cos x - 3 + 5x^3.$

$f(0) = -3$, $f'(x) = 15x^2 + 10\cos x - 10x\sin x$

$f''(x) = 30x - 10\sin x - 10\sin x - 10x\cos x$

$f'''(x) = 30 - 20\cos x - 10\cos x + 10x\sin x.$

$f^{(4)}(x) = +30\sin x + 10\sin x + 10x\cos x$, $f^{(5)}(x) = 50\cos x - 10x\sin x$

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$= -3 + 10x + \frac{50}{5!}x^5 + \cdots$

$\int e^{x^2} dx$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So, $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$

So, $\int e^{x^2} dx = k + x + \frac{x^3}{3} + \frac{x^5}{5.2!} + \frac{x^7}{7.3!} + \cdots$

$$= k + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\, n!}$$

$\frac{3}{2+x} = \frac{3}{2} \frac{1}{1+\frac{x}{2}} = \frac{3}{2}\left(1+\frac{x}{2}\right)^{-1}$

$$= \frac{3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \frac{3}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n$$

For, $\left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$.

$f(x) = \sqrt{x+7}$ around -3.

$f'(x) = \frac{1}{2\sqrt{x+7}}$, $f'(-3) = \frac{1}{4}$

$f''(x) = -\frac{1}{4(x+7)^{3/2}}$

$f'''(x) = \frac{3}{8(x+7)^{5/2}}$

$$T_3(x) = 2 + \frac{1}{4}(x+3) - \frac{1}{64}(x+3)^2 + \frac{1}{512}(x+3)^3$$

$$\# \quad \lim_{x\to 0} \frac{\frac{1}{1+6x^4} - 1 + 6x^4 - 36x^8}{3x^{12}}$$

$$= \lim_{x\to 0} \frac{(1+6x^4)^{-1} - 1 + 6x^4 - 36x^8}{3x^{12}}$$

$$= \lim_{x\to 0} \frac{(\cancel{1} - \cancel{6x^4} + \cancel{(6x^4)^2} - (6x^4)^3 + \cdots) - \cancel{1} + \cancel{6x^4} - \cancel{36x^8}}{3x^{12}}$$

$$= \lim_{x\to 0} \frac{-6^3 x^{12} + 6^4 x^{16} - \cdots}{3x^{12}}$$

$$= \underline{-\,6^3/3}\,.$$