7.3. (1)

Trigonometric Substitution :

$\int 2x\cos x^2\,dx$

$\int f(x)\,dx.$ $\quad x = g(t)$

$dx = g'(t)\,dt.$

$= \int f(g(t))\,g'(t)\,dt.$

→ To make the integrand simpler.!

→ Integrable easily!

$\sqrt{a^2-x^2} \rightarrow x = a\sin\theta$ or $a\cos\theta$ $\quad -\pi/2 \le \theta \le \pi/2$

or $x = a\tanh t$

$\sqrt{a^2+x^2} \rightarrow x = a\tan\theta \quad -\pi/2 < \theta < \pi/2$

$\sqrt{x^2-a^2} \rightarrow x = a\sec\theta, \quad 0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$.

or $x = a\cosh t.$

\# $\int \frac{dx}{\sqrt{x^2-a^2}}$ $\quad x = a\sec\theta$

$dx = a\sec\theta\tan\theta\,d\theta.$

$= \int \frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2(\sec^2\theta-1)}} = \int \sec\theta\,d\theta$

$= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\,d\theta$ $\quad u = \sec\theta+\tan\theta$

$\frac{du}{d\theta} = \sec\theta(\sec\theta+\tan\theta)$

$= \ln|\sec\theta+\tan\theta| + c$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c.$

②

ii) $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$ Let, $x = a\cosh t$.

$dx = a\sinh t\,dt$.

$= \displaystyle\int \frac{\cancel{a}\sinh t}{\cancel{a}\sinh t}\,dt$.

$= t + c = \underline{\cosh^{-1}\left(\frac{x}{a}\right) + c}$

$\displaystyle\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x)^2+2.2x.1+1+1}$

$= \displaystyle\int \frac{dx}{(2x+1)^2+1}$ $2x+1 = \tan\theta$

$2dx = \sec^2\theta\,d\theta$

$= \displaystyle\frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta}$

$= \displaystyle\frac{1}{2}\int d\theta = \frac{1}{2}\theta + c = \underline{\frac{1}{2}\tan^{-1}(2x+1) + c}$

<u>Q.H.W</u>: $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}$

$\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$; $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$; $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

i) $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$ $4-x^2 = u$

$-2x\,dx = du$

$= \displaystyle -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= \displaystyle -\frac{1}{2}\,2\sqrt{u} + c = -\sqrt{u} + c = -\sqrt{4-x^2} + c$.

③

ii) $$\int \frac{dx}{\sqrt{x^2-4}} = \cosh^{-1}\left(\frac{x}{2}\right) + c$$

iii) $$\int \frac{x^2\,dx}{\sqrt{4-x^2}}$$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$= \int \frac{4\sin^2\theta \cdot \cancel{2}\cancel{\cos}\theta\, d\theta}{\cancel{2}\cdot\cancel{\cos}\theta}$$

$$= 4\int \sin^2\theta\, d\theta = 2\int (1-\cos 2\theta)\, d\theta$$

$$= 2\theta - \cancel{2}\,\frac{\sin 2\theta}{\cancel{2}} + c$$

$$= 2\theta - 2\sin\theta\cos\theta + c$$

$$= 2\sin^{-1}(x/2) - x\cdot\sqrt{1-\sin^2\theta} + c$$

$$= 2\sin^{-1}(x/2) - x\sqrt{1-x^2/4} + c$$

$$= 2\sin^{-1}(x/2) - \frac{x}{2}\sqrt{4-x^2} + c$$

\# $$\int \frac{dx}{\sqrt{10+4x-x^2}} = \int \frac{dx}{\sqrt{14-(4-4x+x^2)}}$$

$$= \int \frac{dx}{\sqrt{14-(x-2)^2}}$$

$\left(\sqrt{a^2-x^2}\right)$

$(x-2) = \sqrt{14}\,\sin\theta$

$dx = \sqrt{14}\,\cos\theta\, d\theta$

$$= \int \frac{\sqrt{14}\cos\theta}{\sqrt{14}\cos\theta}\, d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{x-2}{\sqrt{14}}\right) + c$$

$$\int_1^{e} \frac{dy}{y\sqrt{(\ln y)^2+16}} \qquad \text{Let, } \ln y = 4\tan\theta$$

$$\frac{1}{y}\,dy = 4\sec^2\theta\, d\theta$$

$$= \int_0^{\tan^{-1}(\frac{1}{4})} \frac{4\sec^{\cancel{2}}\theta}{\cancel{4}\,\cancel{\sec\theta}}\, d\theta$$

$$= \int_0^{\tan^{-1}(1/4)} \sec\theta\, d\theta = \ln\left|\sec\theta + \tan\theta\right|\Big]_0^{\tan^{-1}(1/4)}$$

$$= \ln\left|\frac{1}{4} + \sec\left(\tan^{-1}\left(\frac{1}{4}\right)\right)\right|$$

$$= \ln\left|\frac{1}{4} + \frac{\sqrt{17}}{4}\right|.$$

\# Q.H.W $\displaystyle\int_1^2 \frac{dx}{x^2\sqrt{4-x^2}}$ (Hint: $x = 2\sin\theta$.)

J.S

16. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$ Take $3x = \sec\theta$

$$= 3^{3}\int_{\sec^{-1}(\sqrt{2})}^{\sec^{-1}(2)} \cos^4\theta\, d\theta \quad \ldots$$

23 $\displaystyle\int \sqrt{5+4x-x^2}\, dx = \int \sqrt{9-(x-2)^2}\, dx$ $\qquad x-2 = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$$= 9\int \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \sin\theta\cos\theta\right] + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\,\frac{x-2}{3\cdot 3}\sqrt{5+4x-x^2} + C$$