Admissible Solution for Hyperbolic Conservation Laws

M.Sc. Project Report

(First Stage)

By

Bankim Chandra Mandal

Roll No: 08509016

Under the Guidance of

Prof. S. Baskar

Department of Mathematics

Indian Institute of Technology, Bombay

November, 2009

Acknowledgments

I would like to thank my guide, **Prof. S.Baskar**, for his support, encouragements, valuable suggestions and overall excellent guidance, without which this report would not have been materialized. It was a very pleasant learning experience for me and I am sure that it will help me a lot in motivating my further studies. I would like to continue my work with my guide later on if I get a suitable opportunity.

Bankim Chandra Mandal

November, 2009

Contents

Chapter 1

Introduction

The Conservation law asserts that, the rate of change of the total amount of substance contained in a fixed domain G is equal to the flux of that substance across the boundary of G.

If u is the density of that substance, f is the flux and η denotes the outward normal to G, dS is the surface element on ∂G, then by the Conseravation law,

$$\frac{d}{dt}\left(\int_G u\,dx\right) = -\int_{\partial G} f \cdot \eta\,dS \tag{1.1}$$

† [The integral in right measures outflow; hence the minus sign is taken]

Applying Divergence theorem, we get:

$$\int_G (u_t + div\,f)\;dx = 0 \tag{1.2}$$

Dividing (1.2) by vol(G) and shrinking G to a point, where all partial derivatives of u and f are continuous, we get:

$$u_t + div\,f = 0$$

✠ We shall see that scalar conservation laws, which we shall define later, have solutions which are not, in general, global classical solutions. Discontinuous solutions for a Cauchy problem can arise either spontaneously due to nonlinearities, or as the result of discontinuities in the initial conditions.

✠ Admissible weak solutions to the scalar conservation equation are unique. At this point, we interpret admissible to mean that the solution satisfies the entropy condition. The selection of the physically relevant solution is based on the so called entropy condition that assert that a shock is formed only when the characteristics carry information toward the shock. We begin by considering the so called Lax entropy condition for selecting the physically relevant weak solution to a problem for a scalar conservation law equation. We shall introduce a definition of a weak entropy solution to the corresponding problem and then we shall prove the existence and uniqueness of the entropy solution for a class of flux functions. The existence property is obtained by regularization of the flux function while for the uniqueness result we shall use Rankine-Hugoniot condition along the line of discontinuity.

✠ To be precise, in chapter 2, we shall discuss about basic notion of Hyperbolic conservation laws, solution procedure of Cauchy problem by Characteristic method, notion of Genuine and Weak solution and at the last, Rankine-Hugoniot conditon. In the next chapter, we shall introduce the notion of various entropy conditions and give a proof of the existence and uniqueness of weak entropy solutions for a particular entropy condition. In the last chapter, we shall discuss about one more entropy condition and prove the equivalence of the entropy conditions for convex flux function.

Chapter 2

Hyperbolic Conservation Law : Basic Theory

In this chapter, we present the general form of systems of Conservation laws in $(k+1)$ space variables. Let, Ω be an open subset of $\mathbb{R}^n$ and let, $f_j : \Omega \to \mathbb{R}^n$; $1 \leq j \leq k$, be k smooth functions.

✠ Consider the system of n equations of Conservation laws:

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{k} \frac{\partial \mathbf{f}_j(\mathbf{u})}{\partial x_j} = 0 \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k, t > 0, \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ is a vector-valued function from $\mathbb{R}^k \times [0, \infty)$ into Ω. The functions $\mathbf{f}_j = \begin{pmatrix} f_{1j} \\ f_{2j} \\ \vdots \\ f_{nj} \end{pmatrix}$; $j = 1, 2, \ldots, k$; are called the flux-functions.

✠ In all the sequel, we shall work with strictly hyperbolic system of conservation laws, which is defined as follows:

Let, $A_j(\mathbf{u}) = \left(\frac{\partial f_{ij}}{\partial u_k} \right)_{1\leq i,k\leq n}$; $j = 1, 2, \ldots, k$, be the Jacobian matrix of $\mathbf{f}_j(\mathbf{u})$. The system (2.1) is called *hyperbolic* if, for any $\mathbf{u} \in \Omega$ and $\mathbf{w} = (w_1, w_2, \ldots, w_k) \in \mathbb{R}^k$, the matrix, $A(\mathbf{u}, \mathbf{w}) = \sum_{j=1}^{k} w_j A_j(\mathbf{u})$ has n real eigenvalues

$$\lambda_1(\mathbf{u}, \mathbf{w}) \leq \lambda_2(\mathbf{u}, \mathbf{w}) \leq \cdots \leq \lambda_n(\mathbf{u}, \mathbf{w})$$

and n linearly independent eigenvectors,

$$r_1(\mathbf{u}, \mathbf{w}), r_2(\mathbf{u}, \mathbf{w}), \cdots, r_n(\mathbf{u}, \mathbf{w})$$

i.e.

$$A(\mathbf{u}, \mathbf{w}) r_j(\mathbf{u}, \mathbf{w}) = \lambda_j(\mathbf{u}, \mathbf{w}) r_j(\mathbf{u}, \mathbf{w}); 1 \leq j \leq n.$$

The eigen-values are also called the wave speeds or *Characteristic speeds* associated with (2.1). Moreover, if the eigenvalues $\lambda_j(\mathbf{u}, \mathbf{w})$ are all distinct, the system (2.1) is called *strictly hyperbolic*. The pair (λ_j, r_j) is referred to as the *j-characteristic field.*

2.1 Single Conservation law

In our discussion, we shall consider strictly hyperbolic sclar conservation law($n = 1$) in one space dimension($k = 1$). So, we have to find a function $u(x, t) : \mathbb{R} \times [0, \infty) \to \Omega$ satisfying:

$$u_t + \frac{\partial}{\partial x}(f(u)) = 0 \tag{2.2}$$

with the initial condition,

$$u(x, 0) = u_0(x), x \in \mathbb{R},$$

where $u_0 : \mathbb{R} \to \Omega$ is given.

2.2 Method of Characteristics:Quasi-linear Equation

From partial differential theory, clearly (2.2) is a quasi-linear equation. The general form of a two dimensional quasi-linear problem is:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{2.3}$$
$$i.e.(a,b,c).(u_x,u_y,-1) = 0$$

Suppose that a solution u is known, and consider the surface graph

$$\varphi(x,y,z) = u(x,y) - z$$

in $\mathbb{R}^3$. A normal vector to this surface is given by,

$$(u_x(x,y), u_y(x,y), -1).$$

As a result, equation (2.3) is equivalent to the geometrical statement that the vector field $(a(x,y,u), b(x,y,u), c(x,y,u))$ is tangent to the surface $\varphi(x,y,z) = u(x,y) - z$ at every point. In other words, the graph of the solution must be a union of integral curves of this vector field. These integral curves are called the characteristic curves of the original partial differential equation.

The equations of the characteristic curve may be expressed invariantly by:

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)},$$

or, if a particular parametrization s of the curves is fixed, then these equations may be written as a system of ordinary differential equations for $x(s), y(s), u(s)$:

$$\left.\begin{array}{rcl} \frac{dx}{ds} & = & a(x,y,u) \\ \frac{dy}{ds} & = & b(x,y,u) \\ \frac{du}{ds} & = & c(x,y,u). \end{array}\right\} \cdots (A)$$

These are the characteristic equations for the original system.

Solving (A) simultaneously, we get two independent integrals:

$$\phi(x, y, u) = c_1 \, , \rho(x, y, u) = c_2$$

So, $F(\phi, \rho) = 0$ is the general solution to (2.3).

Example :-

Consider the quasi-linear equation (Burger's Equation):

$$\left.\begin{aligned} u_t + uu_x = 0 \\ u(x, 0) = u_0(x), x \in \mathbb{R}, \end{aligned}\right\} \cdots (B)$$

Sol: Parameterizing the characteristic equation, we get:

$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{0} = ds \ \cdots (C)$$

with the Cauchy data:

$\Gamma :\ s = 0,\ x = \tau,\ u = u_0(\tau),\ t = 0.\ \cdots (D)$

Solving (C) using (D), we get the solution as:

$$u(x, t) = u_0(x - ut).$$

Remarks: If, in particular, we take, $u_0(x) = -x$, then the sol. will be:

$$u = \frac{x}{(t - 1)}$$

Here, the genuine solution breaks down at $t = 1$.

2.3 General solution : Critical point

Definition:- A genuine(or classical) solution of the partial differential equation (B) in a domain Ω in (x, t)-plane is a function $u(x, t) \in C^1(\Omega)$ which satisfies (B).

✠ A sufficient condition for the existence of a local genuine solution (i.e a sol. valid for $0 < t < t_c$ with some $t_c < \infty$) of the initial value problem (B) is that $u_0(x) \in C^1(\mathbb{R})$.

✠ Consider the function: $G(u, x, t) \equiv u - u_0(x - ut) = 0$
By Implicit Function theorem, the above relation defines a C^1 function $u(x, t)$ if

$$G'_u \neq 0$$

i.e.

$$1 + u'_0(x - ut)t \neq 0$$

Now, $u_x = u'_0(\xi)(1 - tu_x)$; $[\,\xi = x - ut\,]$
i.e. $u_x = \frac{u'_0(\xi)}{1+tu'_0(\xi)}$.

If the initial data is such that $u'_0(\xi) < 0$, $\exists$ a time $t_c > 0$ such that as $t \to t_c-$, the derivative $u_x(x, t) \to -\infty$ and thus the genuine solution cannot be continued beyond $t = t_c$.
The critical time t_c is given by:

$$t_c = -\frac{1}{\min\limits_{\xi \in \mathbb{R}}\{u'_0(\xi)\}} > 0.$$

2.4 Weak Solution and Rankine Hugoniot Condition

Let us go back to the Cauchy problem (2.2). An essential feature of this problem is that, there does not exist , in general, a classical solution beyond some finite time interval, even when the initial condition u_0 is a very smooth function. This lead us to introduce weak solutions of the Cauchy solution (2.2)(which are indeed weaker than the classical sol.)

Definition (Test function):- A Test function is a function $\phi : \mathbb{R} \times \mathbb{R}^+$, which is C^1 with compact support, where, $\text{Supp}(\phi) = \overline{\{\bar{x} \in \mathbb{R} \times \mathbb{R}^+ \ : \ \phi(\bar{x}) \neq 0\}}$.

✠ Weak formulation of Cauchy problem

Multiplying (2.2) by a test function ϕ and integrating over $\mathbb{R} \times \mathbb{R}^+$, we get:

$$\int_{t=0}^{\infty}\int_{-\infty}^{\infty} u_t\phi \, dxdt + \int_{t=0}^{\infty}\int_{-\infty}^{\infty} \frac{\partial f(u)}{\partial x}\,\phi \, dxdt = 0 \tag{2.4}$$

Using the fact that, ϕ has a compact support and using integration by parts, we get from (2.4):

$$\int_{-\infty}^{\infty} u_0(x)\phi(x,0)\, dx + \int_{t=0}^{\infty}\int_{-\infty}^{\infty} (u\phi_t + f(u)\phi_x)\, dxdt = 0 \tag{2.5}$$

(2.5) is called the weak formulation of Cauchy problem (2.2).

Definition (Weak Solution):- Let, $u_0 \in L^{\infty}_{loc}(\mathbb{R})$, where L^{∞}_{loc} is the space of locally bounded measurable functions. A function $u \in L^{\infty}_{loc}(\mathbb{R} \times [0,\infty))$ is called a weak solution of the Cauchy problem (2.2), if $u(x,t) \in \Omega$ a.e. and satisfies (2.5) for any test function ϕ.

Remarks: By construction, a genuine solution of (2.2) is also a weak solution. Moreover, if u happens to be a C^1 function, then it is a classical solution. Indeed, for any test function ϕ, integrating (2.5) by parts,

$$\int_{t=0}^{\infty}\int_{-\infty}^{\infty} (u_t\phi + \frac{\partial f(u)}{\partial x})\,\phi \, dxdt = 0$$

so that, (2.2) holds pointwise.

Next, if we multiply (2.2) by a test function ϕ, after integrating by parts and comparing with (2.5), we obtain:

$$\int_{-\infty}^{\infty} (u(x,0) \ - \ u_0(x))\,\phi(x,0)\, dx = 0$$

which yields the initial condition of Cauchy problem pointwise.

✠ Rankine-Hugoniot Condition

Suppose, u is a piecewise C^1 function. Let, Υ be a curve of discontinuity of u, M be a point of Υ and D be a small ball centered at M.

Let us denote the two open components of D on each side of Υ by D_l and D_r. Then for $\phi \in C_0^\infty(D)$, a C^∞ function having compact support in D, we have:

$$0 = \iint_D \{u\,\phi_t + f(u)\,\phi_x\}\,dxdt = \iint_{D_l} \{u\,\phi_t + f(u)\,\phi_x\}\,dxdt + \iint_{D_r} \{u\,\phi_t + f(u)\,\phi_x\}\,dxdt \tag{2.6}$$

Let, the normal vector $n = (n_1, n_2)$ to the curve Υ points in D_l. Consider,

$$\begin{aligned}
\iint_{D_l} \{u\,\phi_t \,+\, f(u)\,\phi_x\}\,dxdt \;&=\; \iint_{D_l} \frac{\partial}{\partial t}(u\phi)\,dxdt \,+\, \iint_{D_l} \frac{\partial}{\partial x}(f(u)\phi)\,dxdt \,-\, \iint_{D_l} (u_t + \frac{\partial f(u)}{\partial x})\,\phi\,dxdt \\
&=\; \iint_{D_l} (\frac{\partial}{\partial x}, \frac{\partial}{\partial t}).(f(u)\phi, u\phi)\,dxdt \,-\, \iint_{D_l} \{u_t + \frac{\partial}{\partial x} f(u)\}\,\phi\,dxdt
\end{aligned}$$

By Green's theorem,

$$\begin{aligned}
&= \int_{\partial D_l} (f(u_l)\phi, u_l\phi).n\,dS \\
&= \int_{\Upsilon \cap D} \{n_1 f(u_l) + n_2 u_l\}\phi\,dS
\end{aligned}$$

Similarly, considering the integral on D_r (note: $-n$ is the normal vector to the curve Υ points in D_r) from the equation (2.6), we get:

$$0 = \int_{\Upsilon \cap D} \{n_1(f(u_l) - f(u_r)) + n_2(u_l - u_r)\}\phi\,dS \tag{2.7}$$

Since, ϕ is arbitrary, we get:

$$\{n_1(f(u_l) - f(u_r)) + n_2(u_l - u_r)\} = 0$$

Let us set $n \equiv (-\frac{dt}{ds}, \frac{dx}{ds})$ so that, $\frac{n_2}{n_1} = -\frac{dx}{dt}$.
Let, s is the speed of propagation of the discontinuity Υ i.e $s = \frac{dx}{dt}$
Hence,

$$[f(u)] = [u]s \tag{2.8}$$

where, $[f(u)] = f(u_l) - f(u_r)$: jump of $f(u)$ across the curve Υ
and, $[u] = u_l - u_r$: jump of u across the curve Υ.
This relation (2.8) is called the *Rankine-Hugoniot Condition*.

Chapter 3

Mathematical Notion of Entropy

The class of all weak solutions is too wide in the sense that there is no uniqueness for the Cauchy problem:

$$u_t + \frac{\partial}{\partial x}(f(u)) = 0 \tag{3.1}$$

with the initial condition,

$$u(x,0) = u_0(x), x \in \mathbb{R}.$$

For example consider the Burger's equation:

$$u_t + uu_x = 0$$

with the initial value,

$$u_0(x) = \begin{cases} 0 & , if\ x \leq 0 \\ 1 & , if\ x > 0 \end{cases}$$

Sol: A weak sol. is of the form:

$$u(x,t) = \begin{cases} 0 & , if\ x < \frac{t}{2} \\ 1 & , if\ x > \frac{t}{2} \end{cases}$$

Another weak sol. is of the form:

$$u(x,t) = \begin{cases} 0 & , if\ x < 0 \\ \frac{x}{t} & , if\ 0 \leq x \leq t \\ 1 & , if\ x > t \end{cases}$$

In fact, there are infinitely many weak solutions, viz.

$$u(x,t) = \begin{cases} 0 & , if\ x < \frac{\alpha t}{2} \\ \alpha & , if\ \frac{\alpha t}{2} \leq x \leq \alpha t \\ \frac{x}{t} & , if\ \alpha t \leq x \leq t \\ 1 & , if\ x > t \end{cases}$$

where α is any constant satisfying $0 \leq \alpha \leq 1$.

3.1 Introduction to Entropy conditions

As we have just noticed that, a weak sol.to (3.1) is not necessarily unique. Hence, we need to find some criterion, which enables us to choose the "physically relevant" solution among the weak solutions of (3.1). The criterion will be based on the concept of entropy. Let us introduce some of the entropy conditions:

✠ Types of Admissible conditions:

§ Entropy Condition (Version 1):- A discontinuity propagating with speed s, given in R-H condition in previous chapter, satisfied the entropy condition if

$$f'(u_r) < s < f'(u_l)$$

This criterion is known as *Lax's entropy condition.*

§ Entropy Condition (Version 2):- $u(x,t)$ is the entropy solution, if all discontinuities have the property that

$$\frac{f(u) - f(u_r)}{u - u_r} \leq s \leq \frac{f(u) - f(u_l)}{u - u_l}$$

for all u between u_l and u_r.
This condition is due to *Oleinik*.

[NOTE : For convex function, Oleinik entropy condition reduces to condition due to Lax, that I shall try to prove in the last of my discussion.]

3.2 Existence and Uniqueness of Weak solution:Hamilton-Jacobi Method

From now onwards, unless stated otherwise, we shall restrict ourselves to strictly convex flux function f.In this section, we shall study the problem of existence and uniqueness of weak entropy solution of (3.1) (satisfying Lax's entropy condition) using Hamilton-Jacobi theory.

✠ Without loss of generality, we can assume that,

$$f(0) = 0$$

Define

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,dy \tag{3.2}$$

and

$$U_0(x) = \int_{-\infty}^{x} u_0(y)\,dy \tag{3.3}$$

Thus,

$$U_x = u \tag{3.4}$$

Now,

$$\begin{aligned} U_t(x,t) &= \frac{d}{dt}\{\int_{-\infty}^{x} u(y,t)\,dy\} \\ &= \int_{-\infty}^{x} u_t(y,t)\,dy \\ &= -\int_{-\infty}^{x} (f(u))_y\,dy \\ &= f(u(-\infty,t)) - f(u(x,t)) \end{aligned}$$

As, u is integrable and f(0) = 0, using (3.4), we get,

$$\begin{aligned} U_t(x,t) &= -f(u(x,t)) \\ &= -f(U_x). \end{aligned}$$

Hence

$$\left.\begin{aligned} U_t + f(U_x) = 0 \\ U(x,0) = U_0(x) \end{aligned}\right\} \quad \cdots (A)$$

The equation (A) is called the Hamilton-Jacobi equation.

✠ Since f is assumed to be a strictly convex function, $f'' > 0$. Hence, f' is 1-1 and so, $(f')^{-1}$ exists.

Denote

$$G = (f')^{-1}$$

Theorem 3.1.1: [Lax-Oleinik Formula] Let, u be a classical sol.of (3.1).Then,

$$u(x,t) = G\left(\frac{x - y_*}{t}\right) \tag{3.5}$$

where $y_* = y_*(x,t)$ is such that,

$$U(x,t) = U_0(y_*) + t\, g\left(\frac{x - y_*}{t}\right) \tag{3.6}$$

with $g(z) = f'(v)\, v - f(v),\ z = f'(v)$.

Proof :- As f is convex, for any $u \in \mathbb{R}$, we have,

$$\frac{f(u) - f(v)}{u - v} \geq f'(v)$$

where v is arbitrary but fixed.

If u is a classical solution of (3.1), then

$$\begin{aligned} f(U_x) &\geq f(v) + f'(v)\,(U_x - v) \\ \text{i.e. } -U_t &\geq f(v) + f'(v)U_x - f'(v)v \\ \text{i.e. } U_t + f'(v)U_x &\leq f'(v)v - f(v) \end{aligned} \tag{3.7}$$

Now, consider the ordinary first order equation:

$$\frac{dx}{dt} = f'(v); \; x(0) = y \tag{3.8}$$

$$\text{Then } \frac{x-y}{t} = f'(v)$$

$$\text{i.e. } G\left(\frac{x-y}{t}\right) = v \tag{3.9}$$

Integrating (3.7) along the line (3.8), we get,

$$\begin{aligned} \int_0^t \frac{dU}{dt}\,dt &\leq \int_0^t \{f'(v)v - f(v)\}\,dt \\ U(x,t) &\leq U(y,0) + t[f'(v)v - f(v)] \end{aligned}$$

Using the expression of g, we get,

$$U(x,t) \leq U(y,0) + tg\left(\frac{x-y}{t}\right) \tag{3.10}$$

Now, (3.10) is true for all values of y.In particular, it holds for $y = y_*$ as well, for which,

$$\frac{x-y_*}{t} = f'(u)$$

So,

$$u = G\left(\frac{x-y_*}{t}\right) \tag{3.11}$$

It only remains to show that y_* satisfies (3.6). We have,

$$\begin{aligned} U_t + f'(u)U_x &= -f(u) + f'(u)u \\ &= f'(u)u - f(u) \end{aligned}$$

Therefore, the equality holds in (3.7) and so in (3.10).This completes the proof.

[Note: The equation (3.5) is called Lax-Oleinik formula]

Remarks: (1) The weak sol.for the Hamilton-Jacobi equation (A) takes the form:

$$U(x,t) = \min_{y\in\mathbb{R}}\left\{U_0(y) + tg\left(\frac{x-y}{t}\right)\right\} \tag{3.12}$$

$$\text{with } g(z) = f^*(z) = \max_{v\in\mathbb{R}}\{zv - f(v)\} \tag{3.13}$$

Let, $h(v) = vz - f(v)$, for a fixed z. Then $h'(v) = z - f'(v)$ and therefore the maximum value of h is attained at the point v for which $z = f'(v)$. Hence, the g in (3.13) is equivalent to the previous definition.

(2) We have,

$$\begin{aligned}
\frac{dg}{dz} &= \frac{d}{dz}\{f'(v)v - f(v)\} \\
&= \frac{d}{dz}\{f'(G(z))G(z) - f(G(z))\} \\
&= \frac{d}{dz}\{zG(z) - f(G(z))\} \\
&= G(z) + z\frac{d}{dz}G(z) - f'(v)\frac{d}{dz}G(z) \\
&= G(z)
\end{aligned}$$

Hence, we get,

$$\frac{dg}{dz} = G(z) \tag{3.14}$$

Also, we get,

$$g(f'(0)) = f'(0).0 - f(0) = 0 \tag{3.15}$$

Lemma: The mapping $x \mapsto y_*(x,t)$, for a fixed t, is non-decreasing, where y_* is as in Theorem 3.1.1

Proof :- Let, $x_1 < x_2$. Then we have to prove that, $y_1 \leq y_2$, where $y_i = y_*(x_i, t)$ for $i = 1, 2$ Calling the right side of the expression (3.10) as $U(x, y)$, for a fixed t, it is sufficient to show that,

$$U(x_1, y_1) < U(x_2, y), \ \ \forall\, y < y_1$$

Since, at y_1, $U(x_1, y)$ is minimum, so

$$U(x_1, y_1) \leq U(x_1, y) \ \ \forall\, y < y_1 \ \ \cdots (B)$$

`Claim:`

$$\begin{aligned} g\left(\frac{x_2-y_1}{t}\right)+g\left(\frac{x_1-y}{t}\right) &< g\left(\frac{x_1-y_1}{t}\right)+g\left(\frac{x_2-y}{t}\right)\ \forall\, y<y_1 \quad (3.16)\\ \text{Also we have from (3.14), } \frac{dg}{dz} &= G(z)\\ \text{Thus, } g''(z) &= G'(z)>0 \end{aligned}$$

since f' is a strictly increasing and onto function, $G=(f')^{-1}$ is also increasing.

Define, $\lambda = \frac{y_1-y}{x_2+y_1-(x_1+y)}$

Since, $x_1 < x_2$, $0<\lambda<1$ and therefore we have,

$$\begin{aligned} g\left(\frac{x_2-y_1}{t}\right) &= g\left(\lambda\frac{x_1-y_1}{t}+(1-\lambda)\frac{x_2-y}{t}\right)\\ &< \lambda\, g\left(\frac{x_1-y_1}{t}\right)+(1-\lambda)\, g\left(\frac{x_2-y}{t}\right) \qquad (3.17) \end{aligned}$$

Similarly,

$$g\left(\frac{x_1-y}{t}\right)<(1-\lambda)\, g\left(\frac{x_1-y_1}{t}\right)+\lambda\, g\left(\frac{x_2-y}{t}\right) \qquad (3.18)$$

The claim would be justified if we add (3.17) and (3.18).

Now, multiplying (3.16) by t and adding it to (B) yields the required result.

Theorem 3.1.2: [Existence and Uniqueness theorem]

A function u is an entropy sol.for the initial value problem (3.1) if and only if it satisfies the Lax-Oleinik formula (3.5)-(3.6) in the sense of distribution for arbitrary locally integrable initial data u_0.

Moreover, there exists at most one such solution upto a set of measure zero.

Proof :-

`Existence Part:`

First, assume that, u is an entropy solution.We have,

$$\begin{aligned} U_t+f(U_x) &= \int_{-\infty}^{x} u_t(y,t)\,dy+f(u)\\ &= \int_{-\infty}^{x} [u_t(y,t)+(f(u))_y]\,dy \end{aligned}$$

So, u satisfies Hamilton-Jacobi formula and using the same steps as the Theorem 3.1.1, we can prove (3.10), when f is convex. Moreover, since u satisfies the entropy condition, every point (x, t) can be joint to a point y on the line $t = 0$.

Conversely, let, u satisfies Lax-Oleinik formula with u_0 locally integrable.

Then, we have to show that, u is an entropy sol.

Since, u satisfies the Lax-Oleinik formula (3.5)-(3.6), there exists an y_* satisfying (3.6), which is a weak sol. for the Hamilton-Jacobi equation (A). Now, U in (3.6) will be Lipschitz, if U_0 is so. But, since u_0 is bounded, U_0 is Lipschitz and thus, so is U.

Hence, U is differentiable a.e.

We have,

$$U_t + f(U_x) = 0$$

So, for test function ϕ on $\mathbb{R} \times [0, \infty)$, we get,

$$\int_{t=0}^{\infty} \int_{-\infty}^{\infty} \{U_t + f(U_x)\}\phi_x \, dxdt = 0 \tag{3.19}$$

$$\text{Now,} \quad \int_{t=0}^{\infty} \int_{-\infty}^{\infty} U_t \phi_x \, dxdt = -\int_{t=0}^{\infty} \int_{-\infty}^{\infty} U \phi_{tx} \, dxdt - \int_{-\infty}^{\infty} U(x,0)\phi_x(x,0) \, dx$$
$$= \int_{t=0}^{\infty} \int_{-\infty}^{\infty} U_x \phi_t \, dxdt + \int_{-\infty}^{\infty} U_x(x,0)\phi(x,0) \, dx$$

As, $U_0(x) = \int_{-\infty}^{x} u_0(y) \, dy$,

we have: $U_x(x, 0) = u_0(x)$ a.e. So, $\int_{t=0}^{\infty} \int_{-\infty}^{\infty} U_t \phi_x \, dxdt = \int_{t=0}^{\infty} \int_{-\infty}^{\infty} U_x \phi_t \, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0) \, dx$

So, (3.19) gives:

$$\int_{t=0}^{\infty} \int_{-\infty}^{\infty} (u\phi_t + f(u)\phi_x) \, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0) \, dx = 0 \tag{3.20}$$

Which is the weak formulation of Conservation law (3.1).

Now, it is enough to show that, u satisfies the entropy condition.

Let, $x_1 < x_2$

So, $y_1 = y_*(x_1, t) < y_*(x_2, t) = y_2$

i.e. $\frac{x - y_2}{t} \leq \frac{x_1 - y_1}{t}$

Since, G is increasing, we have:

$$G\left(\frac{x - y_2}{t}\right) \leq G\left(\frac{x_1 - y_1}{t}\right)$$

As, u satisfies (3.11), we have:

$$u(x_1, t) = G\left(\frac{x_1 - y_1}{t}\right) \geq G\left(\frac{x_1 - y_2}{t}\right)$$

Now, U is Lipschitz, so

$$\begin{aligned}
|x_2 - x_1| K &\geq |U(x_2,t) - U(x_1,t)| \\
&= \left| \min_{y\in\mathbb{R}} \left\{ U_0(y) + tg\left(\frac{x_2-y}{t}\right) \right\} - \min_{y\in\mathbb{R}} \left\{ U_0(y) + tg\left(\frac{x_1-y}{t}\right) \right\} \right| \\
&\geq |U_0(y_2) + tg\left(\frac{x_2-y_2}{t}\right) - U_0(y_2) - tg\left(\frac{x_1-y_2}{t}\right)|
\end{aligned}$$

$$\begin{aligned}
\text{So,} \quad \left(\frac{x_2-x_1}{t}\right) K &\geq g\left(\frac{x_2-y_2}{t}\right) - g\left(\frac{x_1-y_2}{t}\right) \\
&= \frac{x_2-y_2}{t} G\left(\frac{x_2-y_2}{t}\right) - f\left(G\left(\frac{x_2-y_2}{t}\right)\right) - \frac{x_1-y_2}{t} G\left(\frac{x_1-y_2}{t}\right) \\
&+ f\left(G\left(\frac{x_1-y_2}{t}\right)\right) \\
&\geq \left(\frac{x_2-y_2}{t} - K_F\right) G\left(\frac{x_2-y_2}{t}\right) - \left(\frac{x_1-y_2}{t} - K_F\right) G\left(\frac{x_1-y_2}{t}\right) \quad \cdots (C)
\end{aligned}$$

Now, as any convex function on a bounded interval is Lipschitz, we have,

$$\begin{aligned}
u(x_1,t) = G\left(\frac{x_1-y_1}{t}\right) &\geq G\left(\frac{x_1-y_2}{t}\right) \\
&\geq G\left(\frac{x_2-y_2}{t}\right) - K\left(\frac{x_2-x_1}{t}\right) \\
&= u(x_2,t) - K\left(\frac{x_2-x_1}{t}\right) \\
\text{Thus,} \quad u(x_1,t) - K\frac{x_1}{t} &\geq u(x_2,t) - K\frac{x_2}{t}
\end{aligned}$$

Therefore, the map $x \mapsto u(x,t) - K\frac{x}{t}$ is a non-increasing function. So, it has both left and right hand limits exist at each point.

So, if c be any point on the discontinuity curve Υ, and $x_1 < c < x_2$, then we have,

$$u_r < u_l$$

which is the entropy condition (f being convex)

Hence, u, which satisfies Lax-Oleinik formula, is indeed an entropy weak solution for (3.1).This completes the existence part.

`Uniqueness Part:`

Suppose, u and v are two weak solutions for (3.1).Then,

$$\int_{-\infty}^{\infty} u_0(x)\phi(x,0)\,dx + \int_{t=0}^{\infty}\int_{-\infty}^{\infty} (u\phi_t + f(u)\phi_x)\,dxdt = 0, \qquad (3.21)$$

$$for\ all\ test\ function\ \phi$$

Similarly,

$$\int_{-\infty}^{\infty} u_0(x)\phi(x,0)\,dx + \int_{t=0}^{\infty}\int_{-\infty}^{\infty} (v\phi_t + f(v)\phi_x)\,dxdt = 0 \qquad (3.22)$$

Subtracting (3.22) from (3.21), we get:

$\int_{t=0}^{\infty}\int_{-\infty}^{\infty} \{(u-v)\phi_t + (f(u)-f(v))\phi_x)\}\,dxdt = 0,$

for all test function ϕ

Now,

$$\begin{aligned} f(u)-f(v) &= \int_0^1 \frac{d}{ds} f(su+(1-s)v)\,ds \\ &= \int_0^1 f'(su+(1-s)v)\,(u-v)\,ds \\ \text{So, } f(u(x,t))-f(v(x,t)) &= a(x,t)\{\,u(x,t)-v(x,t)\,\} \\ \text{where, } a(x,t) &= \int_0^1 f'(su+(1-s)v)\,ds \\ \text{Then, } \int_{t=0}^{\infty}\int_{-\infty}^{\infty}(u-v)[\,\phi_t + a(x,t)\phi_x\,]\,dxdt &= 0, \text{ for all test function } \phi \quad (3.23) \end{aligned}$$

Now, suppose, for arbitrary test function ψ, we find ϕ satisfying

$$\phi_t + a(x,t)\phi_x = \psi, \text{ in } \mathbb{R}^2$$

Then, we have from (3.23):

$$\int_{t=0}^{\infty}\int_{-\infty}^{\infty}(u-v)\psi\,dxdt = 0, \quad \forall\psi$$

Hence, u must agree with v. This completes the Uniqueness part.

[Note: While proving uniqueness part, we have used the following result:
for arbitrary test function $\psi \in C^0(\mathbb{R}^2)$, we find ϕ such that $\phi_t + a(x,t)\phi_x = \psi$ holds.]

Chapter 4

A General Notion of entropy condition

Given any smooth solution u of

$$u_t + \frac{\partial}{\partial x}(f(u)) = 0, \tag{4.1}$$

consider

$$\frac{\partial}{\partial t}U(u) + \frac{\partial}{\partial x}F(u) = 0 \tag{4.2}$$

where U and F are sufficiently smooth functions from Ω into $\mathbb{R}$. Also we have,

$$U'(u)f'(u) = F'(u) \tag{4.3}$$

Definition:- Assume that, Ω is a convex set. A convex function $U : \Omega \to \mathbb{R}$ is called an entropy for the system (4.1), if there exist $F : \Omega \to \mathbb{R}$, called *entropy flux* such that (4.3) holds.

Now is the time to state a result before moving to the main result. For time being, we are skipping the proof of the following theorem.

Theorem 4.0.1: Assume that (4.1) admits an entropy U with entropy flux F. Let, $(u_\epsilon)_\epsilon$ be a sequence of sufficiently smooth solution of

$$\frac{\partial u_\epsilon}{\partial t} + \frac{\partial}{\partial x}(f(u_\epsilon)) = \epsilon\, \Delta\, u_\epsilon \tag{4.4}$$

such that,

(a) $\|u_\epsilon\|_\infty \leq K$.

(b) $u_\epsilon \to u$ as $\epsilon \to 0$ $a.e.$

Then, u is a solution of (4.1) and satisfies the entropy condition:

$$\frac{\partial}{\partial t}U(u) + \frac{\partial}{\partial x}F(u) \leq 0$$

in the sense of distributions on $\mathbb{R} \times [0, \infty)$.

Now, it is the time to give the general result. Note that, in this case, u satisfies the jump condition

$$s[U(u)] \geq [F(u)]$$

and it can be proved by arguing exactly as in the derivation of Rankine-Hugoniot condition in chapter 2.

Theorem 4.0.2: Assume that Ω is an interval of $\mathbb{R}$ and $f : \Omega \to \mathbb{R}$ is a strictly convex function. Let, $u : \mathbb{R} \times [0, \infty) \to \Omega$ be a piecewise C^1 function, which satisfies:

$$\begin{aligned} u_t + \frac{\partial}{\partial x}(f(u)) &= 0 \\ \frac{\partial}{\partial t}U(u) + \frac{\partial}{\partial x}F(u) &\leq 0 \end{aligned} \tag{4.5}$$

in the sense of distribution on $\mathbb{R} \times [0, \infty)$, for one strictly convex entropy $U = U_0$. Then, u satisfies (4.5) for any entropy U. This entropy condition is due to Kruzhkov.

Proof :- We have the entropy condition as

$$s\,[U] - [F] \geq 0$$

along a line of discontinuity of u. For fixed u_l, let

$$E_U(v) = \left\{ \frac{f(v) - f(u_l)}{v - u_l} \right\} (U(v) - U(u_l)) - (F(v) - F(u_l))$$

So, we have,

$$E_U(u_r) \geq 0$$

$$\text{and} \;\; E_U(u_l) = 0$$

Let us check that, $v \mapsto E_U(v)$ is a decreasing function, if U is convex; and a strictly decreasing function if U is strictly convex. Let, $v \neq u_l$. Then, we obtain:

$$\begin{aligned} E'_U(v) &= \left\{ \frac{(v-u_l)f'(v) - f(v) + f(u_l)}{(v-u_l)^2} \right\} (U(v) - U(u_l)) + \left\{ \frac{f(v)-f(u_l)}{v-u_l} \right\} U'(v) \\ . & \quad -F'(v) \end{aligned}$$

$$\begin{aligned} \text{Since } U'(v)\, f'(v) &= F'(v), \; we\ have, \\ E'_U(v) &= \left[\frac{f(v)-f(u_l)}{v-u_l} - f'(v) \right] \left[U'(v) - \frac{(U(v)-U(u_l))}{v-u_l} \right] \end{aligned} \tag{4.6}$$

Now, as f is strictly convex,

$$\frac{f(v)-f(u_l)}{v-u_l} > f'(v)$$

and by convexity of U,

$$\frac{U(u_l)-U(v)}{u_l-v} \geq U'(v)$$

So, in virtue of (4.6), we get:

$$E'_U(v) \begin{cases} \leq 0 & , if\ U\ is\ convex \\ < 0 & , if\ U\ is\ strictly\ convex \end{cases}$$

This proves the assumption. This means that, if U_0 is a strictly convex entropy, then,

$$E_{U_0}(u_r) > 0 \text{ if and only if } u_r < u_l$$

and if, U is any convex entropy, then,

$$u_r < u_l \text{ implies } E_U(u_r) \geq 0$$

Hence, if the entropy condition (4.5) holds for one strictly convex entropy U_0, we have for any other convex entropy U,

$$E_{U_0}(u_r) > 0 \;\Rightarrow\; u_r < u_l \;\Rightarrow\; E_U(u_r) \geq 0$$

Hence, the result follows.

Remarks: The above proof shows that, in case of a strictly convex function f, a discontinuity satisfies the entropy condition if and only if

$$u_r < u_l.$$

✠ Now, we shall give a short proof of the equivalence of the Oleinik entropy condition and one due to Lax.

We know by Oleinik entropy condition:

$$\frac{f(u)-f(u_r)}{u-u_r} \leq s \leq \frac{f(u)-f(u_l)}{u-u_l} \tag{4.7}$$

for all u between u_l and u_r. Also, we have from Rankine-Hugoniot condition:

$$[f(u)] = [u]s$$

where $[f(u)] = f(u_l) - f(u_r)$

and, $[u] = u_l - u_r$

So, consider

$$\frac{f(u)-f(u_r)}{u-u_r} \leq \frac{f(u_l)-f(u_r)}{u_l-u_r} \tag{4.8}$$

for any value of u between u_l and u_r.

Now, let $u \to u_r$. Then, (4.8) becomes:

$$f'(u_r) \leq s$$

Similarly, taking limit as $u \to u_l$, we get,

$$f'(u_l) \geq s.$$

Combining these cases, we get:

$$f'(u_r) \leq s \leq f'(u_l) \tag{4.9}$$

The equation (4.9) is nothing but the Lax's entropy condition.

Now, f being convex, $f''(u) \geq 0$, i.e. f' is an increasing function. Hence, from (4.9), we get

$$u_r \leq u_l. \tag{4.10}$$

§ The (geometric) condition (4.7) simply means that the graph of f is below (above, respectively) the line connecting u_l to u_r, when $u_r < u_l$ ($u_l < u_r$, respect.). The condition (4.9) shows that, the characteristic lines impinge on the discontinuity from both sides.

References

[1] Peter D.Lax (1971), *Hyperbolic systems of Conservation Laws and the Mathematical theory of Shock Waves* 2-17, Society for Industrial and Applied Mathematics.

[2] Edwige Godlewski & Pierre-Arnaud Raviart (1991), *Hyperbolic Systems of Conservation laws*, Publication trimestrielle numéro 3/4:300 F.

[3] N.Seguin & J.Vovelle (2003), *Math.Models Methods Appplication Science* 221-257, Analysis and approximation of a scalar Conservation Law with a flux function with discontinuous coefficients.

[4] May, (2003), *Journal of Differential Equation*, Vol.190, issue 1, 131-148.

[5] Brenier Y.(1984), *Averaged Multivalued solution for Scalar Conservation laws*, SIAM Journal on Numerical Analysis (21).

[6] L.C. Evans (1998), *Partial Differential Equations*, Vol.19, Grad.Stud.Math.

[7] T.P.Liu & M.Pierre (1984), *Asymptotic Behavior in Conservation laws*, 419-441.

[8] C.M. Dafermos (1985), *Regularity and large time behaviour of solution of a conservation law*, 201-239, Roy Edinburgh A 99.

[9] Philippe G. LeFloch (2000), *Hyperbolic systems of conservation laws: the theory of classical and Non-classical Shock waves*, 1-33, Birkhäuser.

[10] Liu T.P. (1981), *Admissible solution of Hyperbolic conservation laws*, Amer. Math. Society.

[11] Bressan A. & Philippe G. LeFloch (1997), *Uniqueness of entropy solution for a systems of conservation laws*, 301-331.

[12] Liu T.P. & Yang T. (1999), *Well-posedness theory for Scalar Conservation laws*, Comm. Pure App. Math 52.

[13] Robert C. McOwen (2004), *Partial Differential Equation*, 2nd Edition.

[14] Randall J. LeVeque (1992), *Numerical Methods for Conservation laws*, 19-39, Birkhäuser.

[15] Novi Sad J. (2008) *A Scalar Conservation Law with Nonlinear Diffusion and Linear Dispersion*, Vol.38, No. 3, 129-136.

– –