# Admissible Solution for Hyperbolic Conservation Laws

## M.Sc. Project Report 2

**(Two Stage Combined)**

**By**


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# Chapter 1

# Introduction

The Conservation law asserts that, the rate of change of the total amount of substance contained in a fixed domain $G$ is equal to the flux of that substance across the boundary of $G$.

If $u$ is the density of that substance, $f$ is the flux and $\eta$ denotes the outward normal to $G$, $dS$ is the surface element on $\partial G$, then by the Conseravation law,

$$\frac{d}{dt}\left(\int_G u\,dx\right) = -\int_{\partial G} f\cdot\eta\,dS \tag{1.1}$$

† [ The integral in right measures outflow; hence the minus sign is taken ]

Applying Divergence theorem, we get:

$$\int_G (u_t + div\,f)\ dx = 0 \tag{1.2}$$

Dividing (1.2) by vol($G$) and shrinking $G$ to a point, where all partial derivatives of $u$ and $f$ are continuous, we get:

$$u_t + div\,f = 0$$

✠ We shall see that scalar conservation laws, which we shall define later, have solutions which are not, in general, global classical solutions. Discontinuous solutions for a Cauchy problem can arise either spontaneously due to nonlinearities, or as the result of discontinuities in the initial conditions.

✠ Admissible weak solutions to the scalar conservation equation are unique. At this point, we interpret admissible to mean that the solution satisfies the entropy condition. The selection of the physically relevant solution is based on the so called entropy condition that assert that a shock is formed only when the characteristics carry information toward the shock. In our discussion, we shall prove the existence result by Vanishing Viscosity method and the uniqueness result follow from Kruzkov's Theorem.

✠ To be precise, in chapter 2, we shall discuss the basic theory of Distribution and Sobolev Spaces. In the next chapter, we shall introduce the notion of scalar conservation laws, the Viscous problem and the existence result by Vanishing Viscosity method. In the last chapter, we shall talk about Kruzkov's uniqueness result.

# Chapter 2

# Theory of Distribution and Sobolev Spaces : Basic Theory

In this chapter, we shall introduce some functional spaces which will be of constant use in our discussion. Throughout this note, we shall take $\Omega$ to be an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. Let, $L^2(\Omega)$ be the space of all square integrable functions defined in $\Omega$, i.e

$$L^2(\Omega) := \{v : \int_\Omega |v|^2 \, dx < \infty\}$$

Define a mapping $\langle , \rangle \, : \, L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by

$$\langle v, w \rangle := \int_\Omega v(x)w(x) \, dx$$

**Note :-** The above mapping is not an inner product, as $\langle v, v \rangle = 0$ does not imply $v = 0$.

Example: Consider $\Omega = (0, 1)$ and $v_n(x) = \begin{cases} n, & x = \frac{1}{n} \\ 0, & \text{else} \end{cases}$

Then, $\langle v_n, v_n \rangle = 0$, but $v_n \neq 0$.

✠ To make $L^2(\Omega)$ an inner product space, define

$$v \equiv w \Leftrightarrow \int_\Omega |v|^2\, dx = \int_\Omega |w|^2\, dx.$$

Clearly, "$\equiv$"is an equivalence relation. We call, eventually,

$$\langle v, v\rangle = 0 \Rightarrow v = 0 \text{ a.e.}$$

With this notation, $\langle , \rangle$ defines an inner product on $L^2(\Omega)$ and the induced $L^2$-norm is given by:

$$\|v\| = (\langle v, v\rangle)^{\frac{1}{2}} = \left(\int_\Omega |v|^2\, dx\right)^{\frac{1}{2}}$$

With this norm, $L^2(\Omega)$ becomes a complete inner product space, i.e it is a Hilbert Space.

§ **Definition:-** Define the space of all locally integrable functions in $\Omega$, denoted by $L^1_{loc}(\Omega)$, by:

$$L^1_{loc}(\Omega) = \{v \in L^1(K), \text{ for every compact set } K \text{ with } \overline{K} \subset \Omega\}$$

Example: Note that, $e^x$ is in $L^1_{loc}(0,\infty)$, but not in $L^1(0,\infty)$.

Similarly, when $1 \le p \le \infty$, the space of all $p$-th integrable functions will be denoted by $L^p(\Omega)$. For $1 \le p \le \infty$, the norm is defined by:

$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v|^p\, dx\right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|v\|_{L^\infty(\Omega)} := \text{ess sup}_{x\in\Omega} |v(x)|.$$

## 2.1 Theory of Distributions

In mathematical analysis, distributions (or generalized functions) are objects that generalize functions. Distributions make it possible to differentiate functions whose derivative does not exist in the classical sense. In particular, we shall see that, any locally integrable function has a distributional derivative. Distributions are widely used to formulate generalized solutions of partial differential equations. Where a classical solution may not exist or be very difficult to establish, a distribution solution to a differential equation is often much easier.

§ A multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an $n$-tuple with non-negative integer elements, i.e each $\alpha_i \geq 0$ is integer.

§ By an order of $\alpha$, we mean $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

§ **Definition:-** Define $\alpha$-th order partial derivative of $v$ as:

$$D^{\alpha} v = \frac{\partial^{|\alpha|} v}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \cdots \partial {x_n}^{\alpha_n}}$$

§ Let, $C^m(\Omega) := \{v : D^{\alpha} v \in C(\Omega), |\alpha| \leq m\}$. Similarly, define the space $C^m(\overline{\Omega})$, as $m$-times continuously differentiable functions with bounded and uniformly continuous derivatives up to order $m$ in $\Omega$. And by $C^{\infty}(\Omega)$, we mean a space of infinitely differentiable functions in $\Omega$, i.e $C^{\infty}(\Omega) = \bigcap_{m} C^m(\Omega)$.

§ Define *supp* $v := \overline{\{x \in \Omega : v(x) \neq 0\}}$.

**Note :-** *supp* $v$ is clearly closed. If it is, as well, compact and *supp* $v \subset\subset \Omega$, then $v$ is said to have 'Compact Support'with respect to $\Omega$. Denote by $C_0^{\infty}(\Omega)$, a space of all infinitely differentiable functions with compact support in $\Omega$.

Example : 1. For $\Omega = \mathbb{R}$, consider the function

$$\varphi(x) = \begin{cases} exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

2. For $\Omega = \mathbb{R}^n$, consider the function

$$\phi(x) = \begin{cases} exp\left(\frac{1}{\|x\|^2-1}\right), & \|x\| < 1 \\ 0, & \|x\| \geq 1 \end{cases}$$

Clearly, both $\varphi$, $\phi \in C_0^\infty(\Omega)$.

§ So far, we didn't define any topological structure on $C_0^\infty(\Omega)$. We say $\phi_n \to \phi$ in $C_0^\infty(\Omega)$, if the following conditions are satisfied:
(i) There is a common compact set $K$ in $\Omega$ with $\overline{K} \subset \Omega$ such that $supp\, \phi_n \subset K$.
(ii) $D^\alpha \phi_n \to D^\alpha \phi$ uniformly in $K$, for all multi-indices $\alpha$.

§ **Definition:-** $C_0^\infty(\Omega)$, with such a structure, is denoted by $\mathcal{D}(\Omega)$, the space of test functions.

§ **Definition:-** A continuous linear functional on $\mathcal{D}(\Omega)$ is called a *Distribution*, i.e $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is called a distribution, if $T$ is linear and $T(\phi_n) \to T(\phi)$ as $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$.

§ The space of all distributions will be denoted by $\mathcal{D}'(\Omega)$. This is the topological dual of $\mathcal{D}(\Omega)$.

**Lemma 2.1.1:** $L^1(\Omega) \subset \mathcal{D}'(\Omega)$, i.e every integrable function defines a distribution.
**Proof :-** Assume $f \in L^1(\Omega)$. Then, for $\phi \in \mathcal{D}(\Omega)$, set $T_f(\phi) := \int_\Omega f\phi\, dx$.
Claim: $T_f \in \mathcal{D}'(\Omega)$.
By definition, $T_f$ is linear.
It, thus, suffices to show that, $T_f$ is continuous, i.e as $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, $T_f(\phi_n) \to T_f(\phi)$.

Now,

$$|T_f(\phi_n) - T_f(\phi)| = \left| \int_\Omega f(\phi_n - \phi)\, dx \right| \tag{2.1}$$

As $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, $\exists$ a common compact set $K$ in $\Omega$ such that, $supp\, \phi_n$, $supp\, \phi \subset K$ and $\phi_n \to \phi$ uniformly in $K$.

So, (2.1) yields,

$$\begin{aligned} |T_f(\phi_n) - T_f(\phi)| &\leq \int_K |f(\phi_n - \phi)|\, dx \\ &\leq \max_{x \in K} |\phi_n(x) - \phi(x)| \int_\Omega |f|\, dx \end{aligned}$$

Since, $\int_\Omega |f|\, dx < \infty$ and $\max\limits_{x \in K} |\phi_n(x) - \phi(x)| \to 0$, $T_f(\phi_n) \to T_f(\phi)$ as $n \to \infty$.

Therefore, $T_f$ is continuous and hence $T_f \in \mathcal{D}'(\Omega)$.

Now, if $f, g \in L^1(\Omega)$, with $f = g$ a.e., then,

$T_f(\phi) - T_g(\phi) = \int_\Omega (f - g)\phi\, dx = 0\, \forall \phi \in \mathcal{D}(\Omega)$

So, $T_f = T_g$

Thus, we can identify $f$ as $T_f$ and $T_f(\phi) = \langle f, \phi \rangle$.

Hence, $L^1(\Omega) \subset \mathcal{D}'(\Omega)$.

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## ✠ The Dirac Delta Distribution

§ **Definition:-** For $\mathbf{x} \in \mathbb{R}^n$, define $\delta_x$ by:

$$\delta_x(\phi) = \phi(\mathbf{x}), \ \phi \in \mathcal{D}(\Omega)$$

This, in fact, defines a distribution. Linearity is trivial.

As $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, i.e $\phi_n \to \phi$ uniformly on a common compact support, so

$$\delta_x(\phi_n) = \phi_n(\mathbf{x}) \to \phi(\mathbf{x}) = \delta_x(\phi)$$

So, $\delta_x \in \mathcal{D}'(\Omega)$ and we write

$$\delta_x(\phi) = \langle \delta_x, \phi \rangle = \phi(\mathbf{x})$$

**Note :-** The Dirac delta function can not be generated by locally integrable function. For, if possible, let, $f$ were a locally integrable function such that,

$$\begin{aligned} T_f &= \delta \\ \text{i.e } T_f(\phi) &= \delta(\phi) = \phi(0), \ \text{ for } \phi \in \mathcal{D}(\mathbb{R}^n) \end{aligned}$$

For $\epsilon > 0$, it is possible to find $\phi_\epsilon \in \mathcal{D}(\mathbb{R}^n)$ with $\mathit{supp}\, \phi_\epsilon \subset B_\epsilon(0), 0 \leq \phi_\epsilon \leq 1$ and $\phi_\epsilon \equiv 1$ in $B_{\frac{\epsilon}{2}}(0)$.

Now,

$$\delta(\phi_\epsilon) = 1, \quad \forall \epsilon > 0$$

But, on the other hand,

$$\begin{aligned} \delta(\phi_\epsilon) &= T_f(\phi_\epsilon) \\ &= \int_{\mathbb{R}^n} f\phi_\epsilon \, dx \\ &= \int_{B_\epsilon(\mathbf{0})} f\phi_\epsilon \, dx \\ &\leq \int_{B_\epsilon(\mathbf{0})} |f| \, dx \end{aligned}$$

Since, $f$ is locally integrable, $\int_{B_\epsilon(\mathbf{0})} |f|\, dx \to 0$ as $\epsilon \to 0$.
i.e $\delta(\phi_\epsilon) \to 0$, a contradiction.

§ We have two types of distribution:
(i)**Regular:-** The distribution is said to be *Regular*, if it is generated by a locally integrable function.

(ii)**Singular:-** The distribution is called *Singular*, if it is not regular.
Example:- The Dirac delta function is a singular distribution.

§ **Definition(Distributional Derivative):-** If $T \in \mathcal{D}'(\Omega)$, then define $D(T)$ as

$$(D(T))(\phi) = -\langle T, D(\phi)\rangle, \ \ \forall \phi \in \mathcal{D}(\Omega).$$

§ $D(\phi)$ is simply the derivative of $\phi$. Observe that, $(D(T))$ is a linear map.
For, $\phi_1, \phi_2 \in \mathcal{D}(\Omega)$ and $c \in \mathbb{R}$,

$$\begin{aligned}(D(T))(\phi_1 + c\phi_2) &= -\langle T, D(\phi_1 + \phi_2)\rangle \\ &= -\langle T, D(\phi_1)\rangle - c\,\langle T, D(\phi_2)\rangle \\ &= (D(T))(\phi_1) + c\,(D(T))(\phi_2).\end{aligned}$$

For continuity, we first note that, as $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, then, $D(\phi_n) \to D(\phi)$ in $\mathcal{D}(\Omega)$.
So, $(D(T))(\phi_n) = -\langle T, D(\phi_n)\rangle \to -\langle T, D(\phi)\rangle = (D(T))(\phi)$
Therefore, $D(T) \in \mathcal{D}'(\Omega)$.

§ If $T$ is a distribution, i.e $T \in \mathcal{D}'(\Omega)$, then $\alpha^{th}$ order distributional derivative, say $D^\alpha T \in \mathcal{D}'(\Omega)$, is defined by:

$$\langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \phi\rangle, \ \ \forall \phi \in \mathcal{D}(\Omega).$$

Example: Let, $\Omega = (-1, 1)$ and let, $f(x) = |x|$ in $\Omega$.

Clearly, $f \in \mathcal{D}'(-1, 1)$. For $\phi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}
\langle Df, \phi \rangle &= -\langle f, D\phi \rangle \\
&= -\int_{-1}^{1} f(x)\phi'(x)\,dx \\
&= -\int_{-1}^{0} (-x)\phi'(x)\,dx - \int_{0}^{1} x\phi'(x)\,dx
\end{aligned}$$

Now, $\phi(1) = \phi(-1) = 0$. Hence,

$$\begin{aligned}
\langle Df, \phi \rangle &= x\phi]_{-1}^{0} - \int_{-1}^{0} \phi(x)\,dx - x\phi]_{0}^{1} + \int_{0}^{1} \phi(x)\,dx \\
&= \int_{-1}^{0} (-1)\phi(x)\,dx + \int_{0}^{1} \phi(x)\,dx \\
&= \int_{-1}^{1} H(x)\phi(x)\,dx \\
&= \langle H, \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).
\end{aligned}$$

where,

$$H(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}$$

As $\langle Df, \phi \rangle = \langle H, \phi \rangle$, $\forall \phi \in \mathcal{D}(\Omega)$, so $Df = H$.

$$\begin{aligned}
\langle DH, \phi \rangle &= -\langle H, D\phi \rangle \\
&= -\int_{-1}^{1} H(x)\phi'(x)\,dx \\
&= -\int_{-1}^{0} \phi'(x)\,dx - \int_{0}^{1} \phi'(x)\,dx \\
&= \phi(0) - \phi(-1) - \phi(1) + \phi(0) \\
&= 2\phi(0) = 2\langle \delta, \phi \rangle.
\end{aligned}$$

So, $DH^{\dagger} = 2\delta$ or $D^2 f = 2\delta$

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[1] † In the above discussion we identify $T_H$ as H. And $DH$ is identified as $DT_H$.

## 2.2 Elements of Sobolev Spaces

§ **Definition:-** The *Sobolev Space* $H^1(\Omega)$ is the space of all functions in $L^2(\Omega)$ such that all its partial distributional derivatives are in $L^2(\Omega)$ i.e

$$H^1(\Omega) := \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega), 1 \leq i \leq n\}$$

§ Define a map $\langle , \rangle_1 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by,

$$\langle u, v\rangle_1 = \langle u, v\rangle + \sum_{i=1}^{n} \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\rangle, \quad \forall u, v \in H^1(\Omega)$$

Clearly, $\langle , \rangle_1$ forms an inner product space on $H^1(\Omega)$ and $(H^1(\Omega), \langle , \rangle_1)$ is an inner product space. The induced norm $\|.\|_1$ on $H^1(\Omega)$ is set as:

$$\|u\|_1 = \sqrt{\langle u, u\rangle_1} = \sqrt{\|u\|^2 + \sum_{i=1}^{n} \|\frac{\partial u}{\partial x_i}\|^2}$$

Now, we state the following result regarding the completeness of the Sobolev Space.

**Theorem 2.2.1:** The space $H^1(\Omega)$ with $\|.\|_1$ is a Hilbert space.

§ For positive integer $m$, define the higher order Sobolev Spaces $H^m(\Omega)$ as:

$$H^m(\Omega) := \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), |\alpha| \leq m\}$$

This is again a Hilbert Space with respect to the inner product

$$\langle u, v\rangle_m = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v\rangle$$

and the induced norm:

$$\|u\|_m = \left( \sum_{|\alpha| \leq m} \|D^\alpha u\|^2 \right)^{\frac{1}{2}}$$

§ In general, define Sobolev Space $W^{m,p}(\Omega)$ of order $(m,p)$, $1 \leq p \leq \infty$ by:

$$W^{m,p}(\Omega) := \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), |\alpha| \leq m\}.$$

This is a Banach Space with norm $\|.\|_{m,p}$, where

$$\|u\|_{m,p} = \left( \sum_{|\alpha| \leq m} \int_\Omega \|D^\alpha u\|^p \, dx \right)^{\frac{1}{p}}, \;\; 1 \leq p < \infty$$

and for $p = \infty$,

$$\|u\|_{m,p} = \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

§ **Note :-** (a) When $p = 2$, call $W^{m,2}(\Omega)$ as $H^m(\Omega)$ and its norm $\|.\|_{m,2}$ as $\|.\|_m$

(b) $W^{m,p}(\Omega)$ is Banach Space, with $W^{0,p}(\Omega) = L^p(\Omega)$.

✠ **Remarks:-** $\mathcal{D}(\Omega)$ is not dense in $H^1(\Omega)$. To see this, we claim that, $\mathcal{D}(\Omega)^\perp$ in $H^1(\Omega)$ is not a trivial space.

Assume $u \in \mathcal{D}(\Omega)^\perp$ and $\phi \in \mathcal{D}(\Omega)$.

$$\begin{aligned}
0 &= \langle u, \phi \rangle_1 \\
&= \langle u, \phi \rangle + \sum_{i=1}^n \langle \frac{\partial u}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \rangle \\
&= \langle u, \phi \rangle - \sum_{i=1}^n \langle \frac{\partial^2 u}{\partial x_i^2}, \phi \rangle \\
&= \langle -\Delta u + u, \phi \rangle, \;\; \forall \phi \in \mathcal{D}(\Omega)
\end{aligned}$$

So, $-\Delta u + u = 0$ in $\mathcal{D}(\Omega)$. Clearly, this equation has non-trivial solutions.

Therefore, $\mathcal{D}(\Omega)$ is not dense in $H^1(\Omega)$. Call $H_0^1(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$.

§ **Definition(Dual Space):-** The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$, which consists of all continuous linear functionals on $H_0^1(\Omega)$ with the norm:

$$\|f\|_{-1} = \sup\{|f(v)| : v \in H_0^1(\Omega), \|v\| \leq 1\}$$

**Lemma 2.2.2:** A distribution $f \in H^{-1}(\Omega)$, if and only if there are functions $f_\beta \in L^2(\Omega)$ such that, $f = \sum_{|\alpha| \leq 1} D^\alpha f_\beta$.

✠ **Remarks:-** The Dirac delta function $\delta$ belongs to $H^{-1}(-1,1)$, since there exists Heaviside step function

$$H(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & -1 < x \leq 0 \end{cases}$$

in $L^2(-1,1)$ such that $\delta = DH$.

Similarly, we can define the dual space $(H^1(\Omega))'$ of $H^1(\Omega)$ with the norm defined by:

$$\|f\|_{(H^1(\Omega))'} := \sup\{|f(v)| : v \in H^1(\Omega), \|v\|_1 \leq 1\}.$$

Also, the dual space $H^{-m}(\Omega)$ is defined as the set of all distributions $f$ such that, $\exists$ functions $f_\beta \in L^2(\Omega)$ with $f = \sum_{|\alpha| \leq m} D^\alpha f_\beta$. The norm is given by:

$$\|f\|_{-m} = \sup\{|f(v)| : v \in H_0^m(\Omega), \|v\|_m \leq 1\},$$

$H_0^m(\Omega)$ being the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|.\|_m$.

# Chapter 3

# Scalar Conservation Law:Vanishing Viscosity Method

In this chapter, we shall study the notion of Scalar Conservation law using the notion of Vanishing Viscosity technique. Given $n$ $C^1$ functions $f_j : \mathbb{R} \to \mathbb{R}$, $1 \leq j \leq n$, we consider the scalar conservation law:

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} \frac{\partial f_j(u)}{\partial x_j} = 0 \tag{3.1}$$

where, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $t \in (0, \infty)$.

Set $\mathbf{f}(u) = (f_1(u), f_2(u), \ldots, f_n(u))$ so that, $\operatorname{div} \mathbf{f}(u) = \sum_{j=1}^{n} \frac{\partial f_j(u)}{\partial x_j}$. The equation (3.1) may be written as

$$\frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = 0.$$

Given a function $u_0 : \mathbb{R}^n \to \mathbb{R}$, consider the initial condition as,

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n. \tag{3.2}$$

§ We shall prove that, if $u_0 \in L^\infty(\mathbb{R}^n)$, the cauchy problem $(3.1) - (3.2)$ has a unique entropy solution.

For the existence result, we shall use vanishing viscosity method.

## 3.1 Formulation of Regularization problem

For any $\epsilon > 0$, we associate the initial value problem $(3.1) - (3.2)$ with its parabolic regularization:

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} + \operatorname{div} \mathbf{f}_\epsilon(u_\epsilon) &- \epsilon \Delta u_\epsilon = 0, \quad \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u_\epsilon(\mathbf{x}, 0) &= u_{0\epsilon}(\mathbf{x}). \end{aligned} \tag{3.3}$$

where $\mathbf{f}_\epsilon(u) = (f_{1\epsilon}, f_{2\epsilon}, \ldots, f_{n\epsilon})$ and $u_{0\epsilon}$ are suitable regularizations of the functions $\mathbf{f}$ and $u_0$ respectively. Firstly, we shall study the problems of the form (3.3) and then, pass to the limit as $\epsilon \to 0$.

Now, we shall consider the space of Bounded Variation functions, which will be used frequently later on.

### ✠ Functions of Bounded Variation

$C_0^1(\Omega, \mathbb{R}^n)$ denotes the space of $C^1$ vector-valued functions $\bar{\varphi} : \Omega \to \mathbb{R}^n$ with compact support in $\Omega$.

Setting $\bar{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, consider the norm:

$$\|\bar{\varphi}\|_{L^\infty(\Omega)} := \max_{1 \le i \le n} \left\{ \sup_{\mathbf{x} \in \Omega} |\varphi_i(\mathbf{x})| \right\}.$$

§ Given $g \in L^1_{loc}(\Omega)$, define the total variation of $g$ as:

$$\mathrm{TV}_\Omega(g) := \sup \left\{ \int_\Omega g \operatorname{div} \bar{\varphi} \, dx : \bar{\varphi} \in C_0^1(\Omega, \mathbb{R}^n), \|\bar{\varphi}\|_{L^\infty(\Omega)} \le 1 \right\}$$

§ **Definition:-** A function $g \in L^1_{loc}(\Omega)$ is said to have a *BoundedVariation* in $\Omega$, if $\mathrm{TV}_\Omega(g) < \infty$, i.e

$$\mathrm{BV}(\Omega) := \{g \in L^1_{loc}(\Omega) : \mathrm{TV}_\Omega(g) < \infty\}.$$

§ Now is time to nurture some multi-variate calculus facts.

**Theorem 3.1.1:** (**Gauss-Green Theorem**) Suppose $u \in C^1(\overline{\Omega})$. Then,

$$\int_\Omega u_{x_i}\, d\mathbf{x} = \int_{\partial\Omega} u\nu^i\, dS, \quad (i = 1, 2, \ldots, n)$$

where $\bar{\nu} = (\nu^1, \nu^2, \ldots, \nu^n)$ is the outward unit normal to $\Omega$.

**Theorem 3.1.2:** (**Integration by Parts**) Assume $u, v \in C^1(\overline{\Omega})$. Then,

$$\int_\Omega u_{x_i}\, v\, d\mathbf{x} = -\int_\Omega u\, v_{x_i}\, d\mathbf{x} + \int_{\partial\Omega} uv\, \nu^i\, dS, \quad (i = 1, 2, \ldots, n)$$

**Theorem 3.1.3:** (**Green's Formulas**) Assume $u, v \in C^2(\overline{\Omega})$. Then

$$\begin{aligned} (i) \int_\Omega \Delta u\, d\mathbf{x} &= \int_{\partial\Omega} \frac{\partial u}{\partial \bar{\nu}}\, dS, \\ (ii) \int_\Omega \nabla u . \nabla v\, d\mathbf{x} &= -\int_\Omega v \Delta u\, d\mathbf{x} + \int_{\partial\Omega} \frac{\partial u}{\partial \bar{\nu}}\, v\, dS. \end{aligned}$$

**Lemma 3.1.4:** $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$.

**Proof :-** For $g \in W^{1,1}(\Omega) = \{v \in L^1(\Omega) : \frac{\partial f}{\partial x_i} \in L^1(\Omega), 1 \le i \le n\}$ and $\bar{\varphi} \in C_0^1(\Omega, \mathbb{R}^n)$,

$$\begin{aligned} \int_\Omega g\, \mathrm{div}\bar{\varphi}\, d\mathbf{x} &= \int_\Omega g \sum_{i=1}^n \frac{\partial \varphi_i}{\partial x_i}\, d\mathbf{x} \\ &= \sum_{i=1}^n \int_{\partial\Omega} g\varphi_i \nu^i\, dS - \sum_{i=1}^n \int_\Omega \frac{\partial g}{\partial x_i} \varphi_i\, d\mathbf{x} \\ &= -\int_\Omega \sum_{i=1}^n \frac{\partial g}{\partial x_i} \varphi_i\, d\mathbf{x} \\ &= -\int_\Omega \mathrm{grad}\, g . \bar{\varphi}\, d\mathbf{x} \end{aligned} \tag{3.4}$$

where $\bar{\nu} = (\nu^1, \nu^2, \ldots, \nu^n)$ is the outward unit normal to $\Omega$.

So, $\mathrm{TV}_\Omega(g) \le \int_\Omega |\mathrm{grad}\, g|\, d\mathbf{x} < \infty$.

Hence, $g \in \mathrm{BV}(\Omega)$ and so, $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$.

✠ **Remarks:-** The above inclusion is proper. To see this, consider the function:

$$f(x) = [x], \quad x \in (0, 2).$$

$f$ being a monotone function, it is of Bounded Variation.

Claim: $f \notin W^{1,1}(\Omega)$, in the sense, $T_f \notin W^{1,1}(\Omega)$ We have, for $\phi \in C_0^1(\Omega, \mathbb{R})$,

$$\begin{aligned}
\langle D(T_f), \phi \rangle &= -\langle T_f, \phi' \rangle \\
&= -\int_0^2 f(x)\phi'(x)\,dx \\
&= -\int_1^2 \phi'(x)\,dx \\
&= \phi(1) = \delta_1(\phi)
\end{aligned}$$

Therefore, $D(T_f) = \delta_1$. But, as $\delta_1$ is a singular distribution, $D(T_f)$ does not correspond to an integrable function. So, $f \notin W^{1,1}(\Omega)$.

**Proposition 3.1.5:** Let $\{g_n\}_n$ be a sequence of functions of BV($\Omega$) such that $g_n \to g$ in $L^1_{loc}(\Omega)$. Then we have

$$\mathrm{TV}_\Omega(g) \leq \liminf_{n\to\infty} \mathrm{TV}_\Omega(g_n).$$

**Proof :-** Assume $\bar{\phi} \in C_0^1(\Omega, \mathbb{R}^n)$ such that, $\|\bar{\phi}\|_{L^\infty(\Omega)} \leq 1$.

We have,

$$\int_\Omega g_n \operatorname{div}\bar{\phi}\,d\mathbf{x} \to \int_\Omega g \operatorname{div}\bar{\phi}\,d\mathbf{x}, \quad \text{as } n \to \infty.$$

But, $\int_\Omega g_n \operatorname{div}\bar{\phi}\,d\mathbf{x} \leq \mathrm{TV}_\Omega(g_n)$, and so,

$$\int_\Omega g \operatorname{div}\bar{\phi}\,d\mathbf{x} \leq \liminf_{n\to\infty} \mathrm{TV}_\Omega(g_n).$$

As the above inequality holds for all $\phi \in C_0^1(\Omega, \mathbb{R}^n)$, so,

$$\mathrm{TV}_\Omega(g) \leq \liminf_{n\to\infty} \mathrm{TV}_\Omega(g_n).$$

**Lemma 3.1.6:** The space $L^1(\Omega) \cap \mathrm{BV}(\Omega)$ is a Banach Space for the norm

$$\|g\|_{L_1(\Omega)\cap \mathrm{BV}(\Omega)} = \|g\|_{L_1(\Omega)} + \mathrm{TV}_\Omega(g).$$

**Lemma 3.1.7:** Let, $\Omega$ be a bounded subset of $\mathbb{R}^n$ with a Lipschitz continuous boundary. Then the canonical imbedding of $\mathrm{BV}(\Omega)$ into $L^1(\Omega)$ is compact.

## 3.2 Notion of Measurable Functions

In this section, we shall see some measure theoretic concepts, like weak and strong measurability of a function, special spaces.

§ **Definition(Weakly measurable function):-** If $(X, \mathcal{M})$ is a measurable space and $B$ is a Banach space over a field $\mathbf{K}$ (usually the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$), then $f : X \to B$ is said to be *weakly measurable* if, for every continuous linear functional $g : B \to \mathbf{K}$, the function

$$g \circ f \colon X \to \mathbf{K} \colon x \mapsto g(f(x))$$

is a measurable function with respect to $\mathcal{M}$ and the usual Borel $\sigma$-algebra on $\mathbf{K}$.

§ **Definition(Strongly measurable function):-**

(i) Suppose, $X$ is a Banach space. A function $s : [0, \mathrm{T}] \to X$ is called *Simple*, if it has the form:

$$s(t) = \sum_{i=1}^{m} \mathcal{X}_{E_i}(t) u_i.$$

where, $E_i$'s are Lebesgue measurable subsets of $[0, \mathrm{T}]$ and $u_i \in X$.

(ii) A function $f : [0, \mathrm{T}] \to X$ is said to be *Strongly Measurable*, if there exist a sequence of simple functions $\{s_k\}_k$, $s_j : [0, \mathrm{T}] \to X$, such that, $s_k(t) \to f(t)$ a.e. on $[0, \mathrm{T}]$.

§ Denote by $\mathcal{B}(0,\mathrm{T};X)$, the space of continuous and bounded functions from $[0,\mathrm{T}]$ into $X$ and by $L^p(0,\mathrm{T};X)$, $1 \le p \le \infty$, we mean the space of functions $v : t \in (0,\mathrm{T}) \mapsto v(t) \in X$, which are strongly measurable with respect to the Lebesgue measure and satisfy:

$$\begin{aligned} \|v\|_{L^p(0,\mathrm{T};X)} &= \left(\int_0^{\mathrm{T}} \|v(t)\|_{\mathrm{X}}{}^p \, dt\right)^{\frac{1}{p}} < \infty \ \text{ for } \ 1 \le p < \infty \\ \text{or} \quad \|v\|_{L^\infty(0,\mathrm{T};X)} &= \text{ess sup}_{x\in\Omega} \|v(t)\|_{\mathrm{X}} < \infty \quad \text{for} \ \ p = \infty \end{aligned}$$

Next, we introduce the space

$$W(0,\mathrm{T}) = \{v \in L^2(0,\mathrm{T};H^1(\mathbb{R}^n)) : \frac{\partial v}{\partial t} \in L^2(0,\mathrm{T};H^{-1}(\mathbb{R}^n))\}.$$

## 3.3 The Viscous Problem

Consider the non-linear parabolic problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) \ &- \ a\,\Delta u = 0, \quad \mathbf{x} \in \mathbb{R}^n \ , \ t > 0 \\ u(\mathbf{x},0) \ &= \ u_0(\mathbf{x}) \end{aligned} \tag{3.5}$$

where, $u$ is a function from $\mathbb{R}^n \times [0,\infty)$ to $\mathbb{R}$.

We first prove the existence of a solution of (3.5) by using *fixed point technique*.

**Lemma 3.3.1:** Assume $\mathbf{f} = (f_1, f_2, \dots, f_n)$ is a $C^1$ function, which satisfies the global Lipschitz condition:

$$\|\mathbf{f}(u) - \mathbf{f}(v)\| \le M|u - v|, \quad \forall u, v \in \mathbb{R}.$$

Then, if $u_0 \in L^2(\mathbb{R}^n)$, then the problem (3.5) has a unique solution $u$, which belongs to $W(0,\mathrm{T})$, $\forall \mathrm{T} > 0$.

**Proof :-** Without loss of generality, let, $\mathbf{f}(0) = \mathbf{0}$. Also, let, $\lambda > 0$ be a fixed parameter.

$$\text{Set}, \quad v = u\,exp(-\lambda\,t).$$

Note that, $u$ is a solution of (3.5) if and only if, $v$ is a solution of:

$$\begin{aligned} \frac{\partial v}{\partial t} + \lambda\, v \quad &- \quad a\,\Delta v = -exp(-\lambda\, t)\,\mathrm{div}\,\mathbf{f}(exp(\lambda\, t)\, v) \\ v(\mathbf{x}, 0) \quad &= \quad u_0(\mathbf{x}) \end{aligned} \tag{3.6}$$

Now, let $v \in L^2(\mathbb{R}^n \times (0, \infty)$ be a fixed function. Consider the following linear parabolic problem:

$$\begin{aligned} \frac{\partial w}{\partial t} + \lambda\, w \quad &- \quad a\,\Delta w = -exp(-\lambda\, t)\,\mathrm{div}\,\mathbf{f}\,(exp(\lambda\, t)\, v), \quad \mathbf{x} \in \mathbb{R}^n\,,\ t > 0 \\ w(\mathbf{x}, 0) \quad &= \quad u_0(\mathbf{x}) \end{aligned} \tag{3.7}$$

Now, we have,

$$exp(-\lambda\, t)\, \|\mathbf{f}\,(exp(\lambda\, t)\, v)\| = exp(-\lambda\, t)\, \|\mathbf{f}\,(exp(\lambda\, t)\, v) - \mathbf{f}(0)\| \le M|v|$$

So,

$$exp(-\lambda\, t)\,\mathbf{f}\,(exp(\lambda\, t)\, v) \in L^2(\mathbb{R}^n \times (0, \infty))$$

and therefore,

$$exp(-\lambda\, t)\,\mathrm{div}\,\mathbf{f}\,(exp(\lambda\, t)\, v) \in L^2(0, \infty\,;\ H^{-1}(\mathbb{R}^n)).$$

Now, we shall state a result regarding the existence of solution of a linear parabolic equation.

[ **Result:-** Consider the linear parabolic equation given by:

$$\begin{aligned} \frac{\partial u}{\partial t} + \lambda\, u \quad &- \quad a\,\Delta u = g \\ u(\mathbf{x}, 0) \quad &= \quad u_0(\mathbf{x}) \end{aligned}$$

where, $a > 0$ and $\lambda \in \mathbb{R}$ are given constant. If $u_0 \in L^2(\mathbb{R}^n)$ and $g \in L^2(0, \mathrm{T}\,;\ H^{-1}(\mathbb{R}^n))$, then there exist a unique solution $u \in W(0, \mathrm{T})$ of the above equation.]

So, (3.7) has a unique solution $w \in W(0, \infty)$. Set the mapping

$$\begin{aligned} F_\lambda : L^2(0, \infty\,;\ L^2(\mathbb{R}^n)) \quad &\to \quad W(0, \infty) \quad \text{by} \\ v \quad &\mapsto \quad w = F_\lambda(v) \end{aligned}$$

We shall, now, prove that, for large $\lambda$, $F_\lambda$ is a strict contraction in $L^2(\mathbb{R}^n \times (0,\infty))$. Let, $v_1, v_2 \in L^2(0,\infty\,;\, L^2(\mathbb{R}^n))$ and $w_i = F_\lambda(v_i)$. Suppose, $w = w_1 - w_2$. Therefore, for any $z \in H^1(\mathbb{R}^n$, we get from (3.7),

$$\langle \frac{\partial}{\partial t} w(\cdot,t), z\rangle - a \int_{\mathbb{R}^n} \Delta\, w(\cdot,t).z\, d\mathbf{x} + \lambda \int_{\mathbb{R}^n} w(\cdot,t).z\, d\mathbf{x} \;=\; exp(-\lambda\, t) \int_{\mathbb{R}^n} \{\mathrm{div}\,\mathbf{f}\,(exp(\lambda\, t)\, v_1(\cdot,t)) \;-\; \mathrm{div}\,\mathbf{f}\,(exp(\lambda\, t)\, v_2(\cdot,t))\}.z\, d\mathbf{x}$$

Using Theorem 3.1.2 and Theorem 3.1.3, we get,

$$\langle \frac{\partial}{\partial t} w(\cdot,t), z\rangle + a \int_{\mathbb{R}^n} \mathrm{grad}\, w(\cdot,t).\mathrm{grad}\; z\, d\mathbf{x} \;+\; \lambda \int_{\mathbb{R}^n} w(\cdot,t).z\, d\mathbf{x} = exp(-\lambda\, t) \int_{\mathbb{R}^n} \{\mathbf{f}\,(exp(\lambda\, t)\, v_1(\cdot,t)) \;-\; \mathbf{f}\,(exp(\lambda\, t)\, v_2(\cdot,t))\}.\mathrm{grad}\, z\, d\mathbf{x} \quad (3.8)$$

Let us choose, $z = w(\cdot,t)$ and integrate over $(0,t)$. Using the Green's formula given by:

$$\int_{t_1}^{t_2} \{\langle \frac{\partial}{\partial t} u(\cdot,t), v\rangle + \langle u, \frac{\partial}{\partial t} v(\cdot,t)\rangle\}\, dt = \int_{\mathbb{R}^n} \{(uv)(\mathbf{x},t_2) - (uv)(\mathbf{x},t_1)\}\, d\mathbf{x}, \quad \text{for } \; u,v \in W(0,\mathrm{T})$$

Choosing $u = v = w$ and noting that, $w(\cdot,0) = 0$ in (3.8), we get,

$$\frac{1}{2}\, \|w(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \;+\; \int_0^t \{a\, \|\mathrm{grad}\, w(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} + \lambda\, \|w(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\}\, ds \;\le\; M \int_0^t \|(v_1 - v_2)(\cdot,s)\|_{L^2(\mathbb{R}^n)}\, \|\mathrm{grad}\, w(\cdot,s)\|_{L^2(\mathbb{R}^n)}\, ds$$

Note that, $xy \le ax^2 + \left(\frac{1}{4a}\right)\, y^2$. Taking $x = \|\mathrm{grad}\, w(\cdot,s)\|_{L^2(\mathbb{R}^n)}$ and $y = M\, \|(v_1 - v_2)(\cdot,s)\|_{L^2(\mathbb{R}^n)}$, we get,

$$\frac{1}{2}\, \|w(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \;+\; \lambda \int_0^t \|w(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\, ds \;\le\; \left(\frac{M^2}{4a}\right) \int_0^t \|(v_1 - v_2)(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\, ds$$

As $t \to \infty$, we obtain,

$$\|w_1 - w_2\|^2_{L^2(0,\infty\,;\,L^2(\mathbb{R}^n))} \le \left(\frac{M^2}{4a\lambda}\right)\, \|(v_1 - v_2)(\cdot,s)\|^2_{L^2(0,\infty\,;\,L^2(\mathbb{R}^n))}.$$

Therefore, if $\left(\frac{M^2}{4a\lambda}\right) < 1$, i.e if, $\lambda > \left(\frac{M^2}{4a}\right)$, then $F_\lambda$ becomes a strict contraction mapping and has a unique fixed point $v \in L^2(0,\infty\,;\, L^2(\mathbb{R}^n))$.

Also, $v$ satisfies (3.7) and belongs therefore to $W(0,\infty)$. Finally, observe that, the function $u = exp(\lambda\, t)\, v$ belongs to the space $W(0, \mathrm{T})$, for all $\mathrm{T} > 0$ and is the unique solution to (3.5).

Now, we shall quote two results from S.Kesavan's "Functional Analysis and Applications" as follows:

**Lemma 3.3.2:** Let, $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with $G(0) = 0$. Assume, $\Omega$ to be an open subset of $\mathbb{R}^n$. Then, for all $v \in H^1(\Omega)$, $G \circ v \in H^1(\Omega)$, and,

$$\frac{\partial}{\partial x_i}\,(G \circ v) = (G' \circ v)\,\frac{\partial v}{\partial x_i}, \quad 1 \leq i \leq n.$$

**Lemma 3.3.3:** Let, $\Omega$ be an open subset of $\mathbb{R}^n$. Then, for all $v \in H^1(\mathbb{R}^n)$, $|v| \in H^1(\mathbb{R}^n)$ with

$$\mathrm{grad}\,|v| = \mathrm{sgn} v \;\mathrm{grad}\, v$$

and the mapping $v \mapsto |v|$ is continuous in $H^1(\mathbb{R}^n)$.

**Proof :-** By previous lemma, for $v \in H^1(\mathbb{R}^n)$, $|v| \in H^1(\mathbb{R}^n)$ and $\mathrm{grad}\,|v| = \mathrm{sgn} v \;\mathrm{grad}\, v$, letting $G(v) = |v|$.

Let, $v_n \to v$ in $H^1(\mathbb{R}^n)$. Then, we can extract a subsequence $\{v_{n_p}\}_p$ such that, $v_{n_p}$ converge weakly in $H^1(\mathbb{R}^n)$ to a limit which is necessarily $|v|$ (as $|v_n| \to |v|$ in $L^2(\Omega)$). Also, $\|\,|v_n|\,\|_{H^1(\mathbb{R}^n)} \to \|\,|v|\,\|_{H^1(\mathbb{R}^n)}$. So, the sequence $|v_n|$ converge strongly$^\dagger$ in $H^1(\mathbb{R}^n)$ to $|v|$. Hence, $v \mapsto |v|$ is a continuous function. [1]

[1] † § A sequence of points $\{x_n\}$ in a Hilbert space $H$ is said to converge weakly to a point $x$ in $H$, if $\langle x_n, y\rangle \to \langle x, y\rangle$, for all $y \in H$.

Here, $\langle\cdot,\cdot\rangle$ is understood to be the inner product on the Hilbert space. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

§ Weak convergence is in contrast to strong convergence or convergence in the norm, which is defined by $\|x_n - x\| \to 0$, where $\|x\| = \sqrt{\langle x, x\rangle}$ is the norm of $x$.

**Lemma 3.3.4:** Let, $v \in W(0,T)$. Then, $v_+ = \max\{v, 0\}$ belongs to $L^2(0,T;H^1(\mathbb{R}^n)) \cap \mathcal{B}(0,\mathrm{T};L^2(\mathbb{R}^n))$ and $\forall t_1, t_2 \in [0,T]$,

$$2\int_{t_1}^{t_2} \langle \frac{\partial v}{\partial t}(\cdot,s), v_+(\cdot,s)\rangle\, ds = \|v_+(\cdot,t_2)\|^2_{L^2(\mathbb{R}^n)} - \|v_+(\cdot,t_1)\|^2_{L^2(\mathbb{R}^n)} \tag{3.9}$$

**Theorem 3.3.5:** Assume that, $f$ is a $C^1$ function. If $u_0 \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the problem (3.5) has a unique solution $u$, which belongs to $W(0,T) \cap L^\infty(\mathbb{R}^n \times (0,T))$, for all $T > 0$. Moreover, $u$ satisfies the bound

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \text{ a.e. in } (0,T).$$

**Proof :-** Consider the function $\varphi \in C_0^\infty(\mathbb{R})$ satisfying,

$$\varphi(r) = \begin{cases} 1, & |r| \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \\ 0, & |r| \geq \|u_0\|_{L^\infty(\mathbb{R}^n)} + 1 \end{cases}$$

Set, $g_j = \phi \circ f_j, 1 \leq j \leq n$ and $g = (g_1, g_2, g_3, \ldots, g_n)$.
$g$, being a $C^\infty$ function with a compact support, satisfies Lipschitz condition. By Lemma-3.3.1, there is a unique $u \in W(0,T)$ for any $T > 0$ such that,

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{g}(u) \quad &- \quad a\,\Delta u = 0 \\ u(\mathbf{x},0) \quad &= \quad u_0(\mathbf{x}) \end{aligned} \tag{3.10}$$

Now, we shall prove that, $u$ is indeed a solution of (3.5).
Assume $v = u - \|u_0\|_{L^\infty(\mathbb{R}^n)}$. To show that, $v_+ = 0$.
Since, $|(\epsilon - \epsilon_0)_+| \leq |\epsilon|$ for $\epsilon_0 \geq 0$, we have, $|v_+(\mathbf{x},t)| \leq |u(\mathbf{x},t)|$ and therefore, $v_+(\mathbf{x},t) \in L^2(\mathbb{R}^n)$, for all $t$.
Also, since $u(\cdot,t) \in H^1(\mathbb{R}^n)$ for a.e $t$ and $\operatorname{grad} v(\cdot,t) = \operatorname{grad} u(\cdot,t)$, we have $v(\cdot,t) \in H^1(\mathbb{R}^n)$ a.e and by Lemma-(3.3.3), $v_+(\cdot,t) \in H^1(\mathbb{R}^n)$, for a.e $t$.
Now, we can write

$$\frac{\partial u}{\partial t} - a\,\Delta u = -\operatorname{div} \mathbf{g}(u) = -\sum_{i=1}^{n} g_i'(u) \frac{\partial v}{\partial x_i}$$

Now, for any $z \in H^1(\mathbb{R}^n)$,

$$\langle \frac{\partial}{\partial t} v(\cdot,t), z\rangle - a \int_{\mathbb{R}^n} \Delta\, v(\cdot,t).z\, d\mathbf{x} = -\sum_{i=1}^{n} \int_{\mathbb{R}^n} g_i'(u(\cdot,t)) \frac{\partial}{\partial x_i} v(\cdot,t) z\, d\mathbf{x}$$

Now, by using Theorem-3.1.3, we get after taking $z = v_+(\cdot,t)$,

$$\langle \frac{\partial}{\partial t} v(\cdot,t), v_+(\cdot,t)\rangle + a \int_{\mathbb{R}^n} \operatorname{grad} v(\cdot,t).\operatorname{grad} v_+(\cdot,t)\, d\mathbf{x} = -\sum_{i=1}^{n} \int_{\mathbb{R}^n} g_i'(u(\cdot,t)) \frac{\partial}{\partial x_i} v(\cdot,t) v_+(\cdot,t)\, d\mathbf{x}$$

But, by Lemma-3.3.2,

$$\begin{aligned} \operatorname{grad} v.\operatorname{grad} v_+ &= |\operatorname{grad} v_+|^2 \\ \text{and } \frac{\partial v}{\partial x_i} v_+ &= \frac{\partial v_+}{\partial x_i} v_+ \end{aligned}$$

So, integrating over $(0,t)$ and using Lemma-3.3.4,

$$\begin{aligned} \frac{1}{2}\{\|v_+(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} - \|v_+(\cdot,0)\|^2_{L^2(\mathbb{R}^n)}\} &+ a\int_0^t \|\operatorname{grad} v_+(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\, ds \\ &\leq M \int_0^t \|\operatorname{grad} v_+(\cdot,s)\|_{L^2(\mathbb{R}^n)} \|v_+(\cdot,s)\|_{L^2(\mathbb{R}^n)}\, ds \end{aligned}$$

Since, $v_+(\cdot,0) = 0$, so

$$\|v_+(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{M^2}{2a} \int_0^t \|v_+(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}\, ds$$

[ Using A.M.$\geq$ G.M. inequality, we get $xy \leq ax^2 + \frac{1}{4a} y^2$.
Take $x = \|\operatorname{grad} v_+(\cdot,s)\|_{L^2(\mathbb{R}^n)}, y = M\,\|v_+(\cdot,s)\|_{L^2(\mathbb{R}^n)}$ to get the above inequality.]
Therefore, by Gronwall's Inequality,

$$v_+(\cdot,t) = (u(\cdot,t) - \|u_0\|_{L^\infty(\mathbb{R}^n)})_+ = 0.$$

[ If $U(t) \leq C\,\int_0^t U(s)\, ds$, with $U(t) \geq 0,$ for all $t \geq 0$, then $U(t) = 0$ a.e.
Taking $V(t) = \int_0^t U(s)\, ds$, we get, $\frac{dV}{dt} \leq CV(t)$
i.e $exp(-Ct)\, V(t) \leq 0$, i.e $V(t) = 0$ and so, $U(t) = 0$a.e.]
Similarly, we get,

$$(-u(\cdot,t) - \|u_0\|_{L^\infty(\mathbb{R}^n)})_+ = 0.$$

Therefore, combining the above two result,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \text{ a.e. in } (0,T).$$

So, $g(u) = f(u)$ so that, $u$ is a solution of (3.5) satisfying the required bound.

Now, it suffices to show the uniqueness of this solution. Assume $u_1, u_2$ to be solutions in $W(0,T) \cap L^\infty(\mathbb{R}^n \times (0,T))$. Consider the truncated function $g$ of $f$ on the interval $\{u \in \mathbb{R} : |u| \leq \max_{i=1,2}(\|u_i\|_{L^\infty(\mathbb{R}^n \times (0,T))})\}$. Then, $g$ satisfies a global Lipschitz condition and both $u_1, u_2$ satisfy the equation (3.10). So, by uniqueness result in Lemma-3.3.1, $u_1 = u_2$. In general, we can state the following theorem.

**Theorem 3.3.6:** Suppose $f \in C^m$ and $u_0 \in H^m(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for some integer $m \geq 1$. Then the solution of the problem (3.5) satisfies for all $T > 0$,

$$\begin{aligned} u &\in L^2(0,T; H^{m+1}(\mathbb{R}^n) \cap \mathcal{B}(0,T; H^m(\mathbb{R}^n)) \\ \frac{\partial^k u}{\partial t^k} &\in \begin{cases} L^2(0,T; H^{m+1-2k}(\mathbb{R}^n) \cap \mathcal{B}(0,T; H^{m-2k}(\mathbb{R}^n)), & \text{for } m \geq 2k. \\ L^2(0,T; L^2(\mathbb{R}^n)), & \text{for } m = 2k-1. \end{cases} \end{aligned}$$

§ Now, we shall discuss two results, that will be used to study the properties of the solution. Firstly, we introduce a $C^2$ function $\chi : \mathbb{R}_+ \to \mathbb{R}$ by,

$$\chi(s) = \begin{cases} 1, & 0 \leq s \leq \frac{1}{2} \\ > 0 \text{ and a polynomial}, & \frac{1}{2} \leq s \leq 1 \\ e^{-s}, & s \geq 1 \end{cases}$$

For $R > 0$, define $\varphi_R : \mathbb{R}^n \to \mathbb{R}$ by,

$$\varphi_R(\mathbf{x}) = \chi\left(\frac{|\mathbf{x}|}{R}\right), \ \mathbf{x} \in \mathbb{R}^n.$$

**Lemma 3.3.7:** There exists a constant $C > 0$ such that,

$$\begin{aligned} |\mathrm{grad}\, \varphi_R| &\leq C\, \frac{\varphi_R}{R} \\ \text{and } |\Delta \varphi_R| &\leq C\, \frac{\varphi_R}{R^2} \end{aligned}$$

**Proof :-** We have, $\mathrm{grad}\, \varphi_R(\mathbf{x}) = \frac{1}{R}\, \chi'\left(\frac{|\mathbf{x}|}{R}\right) \frac{\mathbf{x}}{|\mathbf{x}|}$ and

$$\begin{aligned} \Delta \varphi_R(\mathbf{x}) &= \mathrm{div}(\mathrm{grad}\, \varphi_R(\mathbf{x})) \\ &= \mathrm{div}(\phi\, \mathbf{g}), \text{ taking } \phi(\mathbf{x}) = \frac{1}{R}\, \chi'\left(\frac{|\mathbf{x}|}{R}\right) \text{ and } \mathbf{g}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \\ &= \mathrm{grad}\, \phi . \mathbf{g} + \phi\, \mathrm{div}\, \mathbf{g} \\ &= \frac{1}{R}\, \chi'\left(\frac{|\mathbf{x}|}{R}\right) \frac{n-1}{|\mathbf{x}|} + \frac{1}{R^2}\, \chi''\left(\frac{|\mathbf{x}|}{R}\right) \end{aligned}$$

Now, there is a constant $C > 0$ such that,

$$|\chi'(s)| \leq C\,\chi(s)$$
$$|\chi''(s)| \leq C\,\chi(s)$$

Therefore, $|\mathrm{grad}\,\varphi_R| \leq C\,\frac{\varphi_R}{R}$.

Also,

$$|\Delta\varphi_R(\mathbf{x})| \leq \frac{C}{R^2}\,\varphi_R(\mathbf{x}) + \frac{C'}{R\,|\mathbf{x}|}\,\varphi_R(\mathbf{x})$$

So, for $|\mathbf{x}| \geq \frac{R}{2}$, $|\Delta\varphi_R| \leq \frac{d}{R^2}\,\varphi_R(\mathbf{x})$.

and for, $|\mathbf{x}| \leq \frac{R}{2}$, $\Delta\varphi_R(\mathbf{x}) = 0$ and hence the lemma holds trivially in this case.

**Lemma 3.3.8:** For $v, \psi \in H^2(\mathbb{R}^n), \psi \geq 0$,

$$\int_{\mathbb{R}^n} \Delta v\,\mathrm{sgn}(v)\,\psi\,d\mathbf{x} \leq \int_{\mathbb{R}^n} |v|\,\Delta\psi\,d\mathbf{x}.$$

**Proof :-** Consider the piecewise linear continuous function:

$$J_\theta(r) = \begin{cases} -1, & r \leq -\theta \\ \frac{r}{\theta}, & -\theta \leq r \leq \theta, \theta > 0 \\ 1, & r \geq \theta \end{cases}$$

Using Theorem-3.1.3, for $v, \psi \in H^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \Delta v\,J_\theta(v)\,\psi(\mathbf{x})\,d\mathbf{x} = -\int_{\mathbb{R}^n} J'_\theta(v)\,|\mathrm{grad}\,v|^2\,\psi\,d\mathbf{x} - \int_{\mathbb{R}^n} \mathrm{grad}\,v.\mathrm{grad}\,\psi(\mathbf{x})\,J_\theta(v)\,d\mathbf{x}$$

As $J'_\theta \geq 0$ and $\psi \geq 0$, so

$$\int_{\mathbb{R}^n} \Delta v\,J_\theta(v)\,\psi(\mathbf{x})\,d\mathbf{x} \leq -\int_{\mathbb{R}^n} \mathrm{grad}\,v.\mathrm{grad}\,\psi(\mathbf{x})\,J_\theta(v)\,d\mathbf{x}$$

As $\theta \to 0$, we get,

$$\begin{aligned} \int_{\mathbb{R}^n} \Delta v\,\mathrm{sgn}(v)\,\psi\,d\mathbf{x} &\leq -\int_{\mathbb{R}^n} \mathrm{grad}\,|v|.\mathrm{grad}\,\psi\,d\mathbf{x} \\ &\leq \int_{\mathbb{R}^n} |v|\,\Delta\psi\,d\mathbf{x} \end{aligned}$$

Hence the result follows.

**Theorem 3.3.9:** Suppose $f \in C^m$ and $u_0 \in H^m(\mathbb{R}^n) \cap W^{2,1}(\mathbb{R}^n)$, for some integer $m > \frac{n}{2} + 2$. Define

$$M = \sup\{|f'(\omega)| : \omega \in \mathbb{R}, |\omega| \le \|u_0\|_{L^\infty(\mathbb{R}^n)}\}.$$

Then, the solution $u$ of (3.5) satisfies for any $t > 0$:

$(a)$ $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \|u_0\|_{L^\infty(\mathbb{R}^n)}$.

$(b)$ $\displaystyle\int_{\mathbb{R}^n} u(\mathbf{x},t)\,d\mathbf{x} = \int_{\mathbb{R}^n} u_0(\mathbf{x})\,d\mathbf{x}$.

$(c)$ $\displaystyle\|\frac{\partial u}{\partial x_i}(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le \|\frac{\partial u_0}{\partial x_i}\|_{L^1(\mathbb{R}^n)}$.

$(d)$ $\displaystyle\|\frac{\partial u}{\partial t}(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le M\,\|\mathrm{grad}\, u_0\|_{L^1(\mathbb{R}^n)^n} + a\,\|\Delta u_0\|_{L^1(\mathbb{R}^n)}, \forall t \in [0,T]$.

$(e)$ $\|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le \|u_0\|_{L^1(\mathbb{R}^n)} + MT\,\|\mathrm{grad}\, u_0\|_{L^1(\mathbb{R}^n)^n}$.

**Proof :-** We have, $H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, for $\frac{m}{n} > \frac{1}{2}$. So, by Theorem-3.3.6, the solution of (3.5)

$$u \in L^2(0,T;H^{m+1}(\mathbb{R}^n) \cap \mathcal{B}(0,T;H^m(\mathbb{R}^n))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T;H^{m-1}(\mathbb{R}^n) \cap \mathcal{B}(0,T;H^{m-2}(\mathbb{R}^n))$$

and satisfies $(a)$.

We have,

$$\frac{\partial u}{\partial t} + \mathrm{div}\,\mathbf{f}(u) \quad - \quad a\,\Delta u = 0 \tag{3.11}$$

Differentiating with respect to $x_i$,

$$\frac{\partial^2 u}{\partial t \partial x_i} - \frac{\partial}{\partial x_i}\Delta u + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_i} f_j(u) = 0$$

Multiplying both sides by $\mathrm{sgn}(\frac{\partial u}{\partial x_i})$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial x_j \partial x_i} f_j(u)\,\mathrm{sgn}(\frac{\partial u}{\partial x_i}) &= f_j''(u)\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,\mathrm{sgn}(\frac{\partial u}{\partial x_i}) + f_j'(u)\,\frac{\partial^2 u}{\partial x_j \partial x_i}\mathrm{sgn}(\frac{\partial u}{\partial x_i}) \\
&= f_j''(u)\,\frac{\partial u}{\partial x_j}\,\left|\frac{\partial u}{\partial x_i}\right| + f_j'(u)\,\frac{\partial}{\partial x_j}\left|\frac{\partial u}{\partial x_i}\right| \\
&= \frac{\partial}{\partial x_j}\left(f_j'(u)\,\left|\frac{\partial u}{\partial x_i}\right|\right)
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}\left|\frac{\partial u}{\partial x_i}\right| + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(f_j'(u)\left|\frac{\partial u}{\partial x_i}\right|\right) - a\,\Delta\left(\frac{\partial u}{\partial x_i}\right)\operatorname{sgn}(\frac{\partial u}{\partial x_i}) = 0 \tag{3.12}$$

Multiplying (3.12) by $\varphi_R$ and integrate over $\mathbb{R}^n$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|\varphi_R\,d\mathbf{x} &= \int_{\mathbb{R}^n}\mathbf{f}(u)\left|\frac{\partial u}{\partial x_i}\right|.\mathrm{grad}\,\varphi_R\,d\mathbf{x}\\ &+ a\int_{\mathbb{R}^n}\Delta\left(\frac{\partial u}{\partial x_i}\right)\operatorname{sgn}(\frac{\partial u}{\partial x_i})\,\varphi_R\,d\mathbf{x}\end{aligned} \tag{3.13}$$

Taking in $v = \frac{\partial u}{\partial x_i}$ and $\psi = \varphi_R$ Lemma-3.3.8, we get,

$$\int_{\mathbb{R}^n}\Delta\left(\frac{\partial u}{\partial x_i}\right)\operatorname{sgn}(\frac{\partial u}{\partial x_i})\,\varphi_R\,d\mathbf{x} \le \int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|\Delta\varphi_R\,d\mathbf{x}$$

So, from (3.13),

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|\varphi_R\,d\mathbf{x} &\le \int_{\mathbb{R}^n}\mathbf{f}(u).\mathrm{grad}\,\varphi_R\left|\frac{\partial u}{\partial x_i}\right|d\mathbf{x}\\ &+ a\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|\Delta\varphi_R\,d\mathbf{x}\end{aligned}$$

Thus, using Lemma-3.3.7, we get,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|\varphi_R\,d\mathbf{x} \le \frac{C}{R}\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}\right|\varphi_R\,d\mathbf{x}$$

Integrating over $[0,t]$,

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(\cdot,t)\right|\varphi_R\,d\mathbf{x} &- \int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(\cdot,0)\right|\varphi_R\,d\mathbf{x}\\ &\le \frac{C}{R}\int_0^t\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(\cdot,s)\right|\varphi_R\,d\mathbf{x}\,ds\end{aligned}$$

Hence, by generalized Gronwall's inequality$^\dagger$,

$$\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(\cdot,t)\right|\varphi_R\,d\mathbf{x} \le \left(1 + \frac{C}{R}\,t\,exp(\frac{Ct}{R})\right)\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(\cdot,0)\right|\varphi_R\,d\mathbf{x}$$

2

[2]† Let, $U(t)$ be a nonnegative, integrable function on $[0,T]$, for which

$$U(t) \le C_1\int_0^t U(s)\,ds + C_2.$$

for constants $C_1, C_2 \ge 0$. Then,

$$U(t) \le C_2\,\{1 + C_1\,t\,exp(C_1 t)\}$$

a.e in $[0,T]$.

Now, as $R \to \infty$, we have,

$$\int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial x_i}(\cdot, t) \right| \, d\mathbf{x} \leq \int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial x_i}(\cdot, 0) \right| \, d\mathbf{x},$$

which proves $(c)$.

Similarly, multiplying by $\mathrm{sgn}(\frac{\partial u}{\partial t})\, \varphi_R$ followed by differentiating (3.11) with respect to $t$ and arguing as above, we get,

$$\| \frac{\partial u}{\partial t}(\cdot, t) \|_{L^1(\mathbb{R}^n)} \leq \| \frac{\partial u}{\partial t}(\cdot, 0) \|_{L^1(\mathbb{R}^n)}$$

Now,

$$\begin{aligned} \frac{\partial u}{\partial t}(\cdot, 0) &= -\operatorname{div} \mathbf{f}(u_0) + a\, \Delta u_0 \\ &= -\sum_{j=1}^{n} f_j'(u_0)\, \frac{\partial u_0}{\partial x_j} + a\, \Delta u_0 \end{aligned}$$

Thus, we get,

$$\| \frac{\partial u}{\partial t}(\cdot, 0) \|_{L^1(\mathbb{R}^n)} \leq M \, \| \operatorname{grad} u_0 \|_{L^1(\mathbb{R}^n)^n} + a \, \| \Delta u_0 \|_{L^1(\mathbb{R}^n)},$$

which proves $(d)$.

Multiplying (3.11) by $\mathrm{sgn}(u)\, \varphi_R$ and integrating over $\mathbb{R}^n$,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{R}^n} |u| \, \varphi_R \, d\mathbf{x} \;\; &+ \;\; \sum_{j=1}^{n} \int_{\mathbb{R}^n} f_j'(u) \, \frac{\partial u}{\partial x_j} \mathrm{sgn}(u) \, \varphi_R \, d\mathbf{x} \\ &- \;\; a \int_{\mathbb{R}^n} \Delta u \, \mathrm{sgn}(u) \, \varphi_R \, d\mathbf{x} = 0 \end{aligned}$$

Using Lemma-3.3.8 and Lemma-3.3.7, we get,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{R}^n} |u| \, \varphi_R \, d\mathbf{x} &\leq a \int_{\mathbb{R}^n} |u| \, \Delta \varphi_R \, d\mathbf{x} + M \int_{\mathbb{R}^n} |\operatorname{grad} u| \varphi_R \, d\mathbf{x} \\ &\leq \frac{aC}{R^2} \int_{\mathbb{R}^n} |u| \, \varphi_R \, d\mathbf{x} + M \int_{\mathbb{R}^n} |\operatorname{grad} u| \varphi_R \, d\mathbf{x} \end{aligned}$$

Integrate with respect to $t$ and letting $R \to \infty$, we get,

$$\| u(\cdot, t) \|_{L^1(\mathbb{R}^n)} \leq \| u_0 \|_{L^1(\mathbb{R}^n)} + M \int_0^t \| \operatorname{grad} u(\cdot, s) \|_{L^1(\mathbb{R}^n)} \, ds$$

Also,

$$\| \frac{\partial u}{\partial x_i} \|_{L^1(\mathbb{R}^n)} \leq \| \frac{\partial u_0}{\partial x_i} \|_{L^1(\mathbb{R}^n)}$$

Thus, we get,

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le \|u_0\|_{L^1(\mathbb{R}^n)} + MT\,\|\text{grad}\, u_0\|_{L^1(\mathbb{R}^n)^n}.$$

which proves $(e)$.

Now, it remains to prove $(b)$. Multiply (3.11) by $\varphi_R$ and integrate over $\mathbb{R}^n$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^n} u\,\varphi_R\,d\mathbf{x} \;&-\; a\int_{\mathbb{R}^n} \Delta u\,\varphi_R\,d\mathbf{x}\\ &+\; \sum_{j=1}^{n}\int_{\mathbb{R}^n} f_j'(u)\,\frac{\partial u}{\partial x_j}\,\varphi_R\,d\mathbf{x} = 0\\ \frac{\partial}{\partial t}\int_{\mathbb{R}^n} u\,\varphi_R\,d\mathbf{x} \;&+\; a\int_{\mathbb{R}^n} \nabla u\,\nabla\varphi_R\,d\mathbf{x}\\ &-\; \int_{\mathbb{R}^n} \mathbf{f}(u).\text{grad}\,\varphi_R\,d\mathbf{x} = 0\end{aligned}$$

So, we get,

$$\left|\frac{\partial}{\partial t}\int_{\mathbb{R}^n} u\,\varphi_R\,d\mathbf{x}\right| \le \frac{C}{R}\int_{\mathbb{R}^n} |\mathbf{f}(u)|\,\varphi_R\,d\mathbf{x} + \frac{aC}{R}\int_{\mathbb{R}^n} |\nabla u|\,d\mathbf{x}$$

Now, as $R \to \infty$, the R.H.S in the above inequality tends to 0. Thus,

$$\int_{\mathbb{R}^n} u(\mathbf{x},t)\,d\mathbf{x} = \int_{\mathbb{R}^n} u_0(\mathbf{x})\,d\mathbf{x}.$$

This completes the result.

## 3.4 Existence of an Entropy Solution

We use the vanishing viscosity method in order to prove the existence of an entropy solution $u$. Assume $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a function such that,

$(i)$ $\zeta(\mathbf{x}) \ge 0$ and its support is contained in the unit ball of $\mathbb{R}^n$.

$(ii)$ $\displaystyle\int_{\mathbb{R}^n} \zeta(\mathbf{x})\,d\mathbf{x} = 1.$

$(iii)$ $\zeta(-\mathbf{x}) = \zeta(\mathbf{x}).$

For all $\epsilon > 0$, set

$$\zeta_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^n}\,\zeta\left(\frac{\mathbf{x}}{\epsilon}\right).$$

Assume, $u_{0\epsilon} = u_0 * \zeta_\epsilon$ and $f_{j\epsilon} = f_j * \zeta_\epsilon$. Then, $\mathbf{f}_\epsilon = (f_{1\epsilon}, f_{2\epsilon}, \ldots, f_{n\epsilon})$ is a $C^\infty$ function.

**Lemma 3.4.1:** Assume $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n)$. Then, $u_{0\epsilon} \in H^m(\mathbb{R}^n)$ for all $m$ and satisfies the bounds:

$$\begin{array}{ll} (a) & \|u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)} \\ (b) & \displaystyle\int_{\mathbb{R}^n} u_{0\epsilon}(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} u_0(\mathbf{x})\, d\mathbf{x}. \\ (c) & \|u_{0\epsilon}\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \\ (d) & \|\mathrm{grad}\, u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} \leq \mathrm{TV}(u_0) \\ (e) & \|\Delta u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} \leq \dfrac{C}{\epsilon}\, \mathrm{TV}(u_0) \end{array}$$

for some constant $C$.

**Lemma 3.4.2:** Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n)$. Then, the regularized problem (3.3) has a unique $C^\infty$ solution, which satisfies for any $t \geq 0$,

$$\begin{array}{ll} (a) & \|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \\ (b) & \displaystyle\int_{\mathbb{R}^n} u_\epsilon(\mathbf{x},t)\, d\mathbf{x} = \int_{\mathbb{R}^n} u_0(\mathbf{x})\, d\mathbf{x}. \\ (c) & \|\mathrm{grad}\, u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq \mathrm{TV}(u_0) \\ (d) & \|\dfrac{\partial u_\epsilon}{\partial t}(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq C\, \mathrm{TV}(u_0) \\ (e) & \|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)} + CT\, \mathrm{TV}(u_0) \end{array}$$

**Proof :-** Since by previous lemma, $u_{0\epsilon} \in H^m(\mathbb{R}^n)$, for all $m \geq 0$ and $\mathbf{f}_\epsilon$ is a $C^\infty$ function, we may apply Theorem-3.3.6. So, the problem (3.3) has a unique solution $u_\epsilon$ such that,

$$\frac{\partial^k u}{\partial t^k} \in \mathcal{B}(0,T; H^{m-2k}(\mathbb{R}^n)).$$

Hence, $u_\epsilon$ is a $C^\infty$ function. Now, appeal to Theorem-3.3.9 along with Lemma-3.4.1 to get

$$\begin{aligned} \|u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\leq \|u_{0\epsilon}\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)}. \\ \int_{\mathbb{R}^n} u_\epsilon(\mathbf{x},t)\, d\mathbf{x} &= \int_{\mathbb{R}^n} u_{0\epsilon}(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} u_0(\mathbf{x})\, d\mathbf{x}. \\ \|\mathrm{grad}\, u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\leq \|\mathrm{grad}\, u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} \leq \mathrm{TV}(u_0) \\ \text{and } \|\frac{\partial u_\epsilon}{\partial t}(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\leq M_\epsilon\, \|\mathrm{grad}\, u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} + \epsilon\, \|\Delta u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} \\ &\leq M_\epsilon\, \mathrm{TV}(u_0) + C_1\, \mathrm{TV}(u_0) \end{aligned}$$

where, $M_\epsilon = \sup\{|\mathbf{f}'_\epsilon(s)| : |s| \le \|u_{0\epsilon}\|_{L^\infty(\mathbb{R}^n)}\}$. Since,

$$f'_{j\epsilon}(s) = \int_{\mathbb{R}} f'_j(s-\theta)\,\zeta_\epsilon(\theta)\,d\theta,$$

we get, $M_\epsilon \le \sup\{|\mathbf{f}'(s)| : |s| \le \|u_0\|_{L^\infty(\mathbb{R}^n)} + \epsilon\} \le C_2$
so that,

$$\|\frac{\partial u_\epsilon}{\partial t}(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le (C_1 + C_2)\,\mathrm{TV}(u_0).$$

Lastly, for $0 \le t \le T$,

$$\begin{aligned}\|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le \|u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} + M_\epsilon\,T\,\|\mathrm{grad}\,u_{0\epsilon}\|_{L^1(\mathbb{R}^n)} \\ &\le \|u_0\|_{L^1(\mathbb{R}^n)} + C_2\,T\,\mathrm{TV}(u_0)\end{aligned}$$

This completes the lemma.

## ✠ A General notion of Entropy

§ Given any smooth solution $u$ of

$$u_t + \mathrm{div}\,(\mathbf{f}(u)) = 0, \tag{3.14}$$

consider

$$\frac{\partial}{\partial t}U(u) + \sum_{j=1}^{n}\frac{\partial}{\partial x_j}F_j(u) = 0 \tag{3.15}$$

where $U$ and $F_j$ are sufficiently smooth functions from $\mathbb{R}$ into $\mathbb{R}$. Also we have,

$$U'(u)f'_j(u) = F'_j(u), 1 \le j \le n. \tag{3.16}$$

**Definition:-** Assume that, $\Omega$ is a convex subset of $\mathbb{R}$. A convex function $U : \Omega \to \mathbb{R}$ is called an entropy for the system (3.14), if there exist $F_j : \Omega \to \mathbb{R}$, called *entropy flux* such that (3.16) holds.

**Theorem 3.4.3:** Assume that (3.1)-(3.2) admits an entropy $U$ with entropy fluxes $F_j, 1 \le j \le n$. Let, $\{u_\epsilon\}_\epsilon$ be a sequence of sufficiently smooth solution of (3.3) such that,

$(a)$ $\|u_\epsilon\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} \le K$.

$(b)$ $u_\epsilon \to u$ as $\epsilon \to 0$ a.e. in $\mathbb{R}^n \times (0,\infty)$

Then, $u$ is a solution of (3.1)-(3.2) and satisfies the entropy condition:

$$\frac{\partial}{\partial t}U(u) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} F_j(u) \leq 0$$

in the sense of distributions on $\mathbb{R}^n \times (0, \infty)$.

**Theorem 3.4.4:** Assume $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n)$ and $\mathbf{f}$ is a $C^1$ function. Then, the given problem (3.1)-(3.2) has an entropy solution $u$.
Moreover, we have $u \in L^\infty(\mathbb{R}^n \times (0,\infty)) \cap \mathcal{B}(0, T; L^1(\mathbb{R}^n))$ for any $T > 0$ and $u(\cdot, t) \in \mathrm{BV}(\mathbb{R}^n)$ for all $t > 0$ with

$$\begin{aligned}
&(a) \quad \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \text{ a.e} \\
&(b) \quad \mathrm{TV}(u(\cdot,t)) \leq \mathrm{TV}(u_0) \\
&(c) \quad \int_{\mathbb{R}^n} |u(\mathbf{x}, t_2) - u(\mathbf{x}, t_1)|\, d\mathbf{x} \leq C\, \mathrm{TV}(u_0)\, |t_2 - t_1|, \quad \text{for } t_1, t_2 \geq 0
\end{aligned}$$

**Proof :-** Consider a sequence of solutions $\{u_\epsilon\}_\epsilon$ constructed by the vanishing viscosity method of the equation (3.3). By Lemma-3.4.2, the sequence $\{u_\epsilon\}_\epsilon$ is bounded in $L^\infty(\mathbb{R}^n \times (0,\infty)) \cap W^{1,1}_{loc}(\mathbb{R}^n \times (0,\infty))$. Let, $\{K_n\}_n$ be a countable increasing sequence of compact subsets of $\mathbb{R}^n \times [0,\infty)$, which cover $\mathbb{R}^n \times [0,\infty)$. Now, taking Theorem-3.1.7 into account, we can extract a subsequence, from the sequence $\{u_\epsilon\}_\epsilon$, converging almost everywhere in $L^1(K_n)$, for each $n$. So, using the diagonal extraction procedure, we obtain a subsequence, still denoted by $\{u_\epsilon\}_\epsilon$, such that

$$\|u_\epsilon\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)}$$

and

$$\begin{aligned}
u_\epsilon &\rightarrow u \text{ in } L^1_{loc}(\mathbb{R}^n \times (0,\infty)) \\
u_\epsilon &\rightarrow u \text{ in } \mathbb{R}^n \times (0,\infty)
\end{aligned}$$

Now, we check that, the limit $u$ is an entropy solution of the given problem (3.1)-(3.2). We have $u_{0\epsilon} \rightarrow u_0$ in $L^1(\mathbb{R}^n)$ and $\mathbf{f}_\epsilon \rightarrow \mathbf{f}$ uniformly on compact subset of $\mathbb{R}$ so that

$$\|\mathbf{f}_\epsilon(u_\epsilon) - \mathbf{f}(u_\epsilon)\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} \rightarrow 0$$

Since, $\mathbf{f}(u_\epsilon) \to \mathbf{f}(u)$ in $L^1_{loc}(\mathbb{R}^n \times (0,\infty))^n$ a.e. we obtain

$$\mathbf{f}_\epsilon(u_\epsilon) \to \mathbf{f}(u) \text{ in } L^1_{loc}(\mathbb{R}^n \times (0,\infty))^n.$$

Hence, arguing as Theorem-3.4.3, we conclude that, $u$ is an entropy weak solution of (3.1)-(3.2), which satisfies $(a)$.

`Claim:` $u_\epsilon \to u$ in $\mathcal{B}(0,T; L^1_{loc}(\mathbb{R}^n))$ for any $T > 0$. (A)

It follows from Lemma-3.4.2 that, $u_\epsilon$ remains in a bounded set of $\mathcal{B}(0,T; W^{1,1}(\mathbb{R}^n))$ and $\frac{\partial u_\epsilon}{\partial t}$ remains in a bounded set of $\mathcal{B}(0,T; L^1(\mathbb{R}^n))$.

Let, $\Omega$ be a bounded subset of $\mathbb{R}^n$ with a smooth boundary. Now, for $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} u_\epsilon(\cdot,t_2) - u_\epsilon(\cdot,t_1) &= \int_{t_1}^{t_2} \frac{\partial u_\epsilon}{\partial t}(\cdot,s)\,ds \\ \text{so that } \|u_\epsilon(\cdot,t_2) - u_\epsilon(\cdot,t_1)\|_{L^1(\Omega)} &\le \int_{t_1}^{t_2} \|\frac{\partial u_\epsilon}{\partial t}\|_{L^1(\Omega)}\,ds \\ \text{and } \|u_\epsilon(\cdot,t_2) - u_\epsilon(\cdot,t_1)\|_{L^1(\Omega)} &\le C\,\mathrm{TV}(u_0)\,|t_2 - t_1| \end{aligned} \tag{3.17}$$

Hence, the sequence $\{u_\epsilon\}_\epsilon$ is uniformly equicontinuous from $[0,T]$ in $L^1(\Omega)$. Since the canonical imbedding from $W^{1,1}(\Omega)$ into $L^1(\Omega)$ is compact, $\{u_\epsilon(\cdot,t)\}_\epsilon$ remains in a compact subset of $L^1(\Omega)$.

Now, apply Arzela Ascoli's Theorem, which says that, every bounded and equicontinuous sequence in $C(X)$ has a uniformly convergent subsequence, to extract a subsequence from the sequence $\{u_\epsilon\}_\epsilon$, still denoted by $\{u_\epsilon\}_\epsilon$ such that

$$u_\epsilon \to u \text{ in } \mathcal{B}(0,T; L^1(\Omega)).$$

By using a collection of compact subsets of $\mathbb{R}^n$ and the diagonal extraction procedure again, we get,

$$u_\epsilon \to u \text{ in } \mathcal{B}(0,T; L^1_{loc}(\mathbb{R}^n))$$

`Claim:` $u \in \mathcal{B}(0,T; L^1(\mathbb{R}^n))$.

Given any bounded subset $\Omega$ of $\mathbb{R}^n$ and for any $t \in [0,T]$, it follows from Lemma-3.4.2 that,

$$\|u_\epsilon(\cdot,t)\|_{L^1(\Omega)} \le C_0 = \|u_0\|_{L^1(\mathbb{R}^n)} + CT\,\mathrm{TV}(u_0)$$

and so,

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le C_0.$$

Since it holds for any $\Omega$, we get that, $u(\cdot,t) \in L^1(\mathbb{R}^n)$. Similarly, it follows from (3.14) that,

$$\|u(\cdot,t_2) - u(\cdot,t_1)\|_{L^1(\mathbb{R}^n)} \leq C\,\mathrm{TV}(u_0)\,|t_2 - t_1|$$

which proves our claim as well as $(c)$.

Lastly, assume $\phi \in C_0^1(\mathbb{R}^n), \|\phi\|_{L^\infty(\mathbb{R}^n)} \leq 1$. Then by part$(c)$ of Lemma-3.4.2,

$$\begin{aligned} \int_{\mathbb{R}^n} u_\epsilon(\cdot,t)\,\mathrm{div}\,\phi\,d\mathbf{x} &= -\int_{\mathbb{R}^n} \mathrm{grad}\,u_\epsilon(\cdot,t).\phi\,d\mathbf{x} \\ &\leq \mathrm{TV}(u_0)\,\|\phi\|_{L^\infty(\mathbb{R}^n)} \\ \text{using (A)} \quad \int_{\mathbb{R}^n} u(\cdot,t)\,\mathrm{div}\,\phi\,d\mathbf{x} &\leq \mathrm{TV}(u_0), \forall t \geq 0 \end{aligned}$$

Hence,

$$\mathrm{TV}(u(\cdot,t)) \leq \mathrm{TV}(u_0), \forall t \geq 0.$$

# Chapter 4

# Kruzkov's Uniqueness Result

We now turn to the uniqueness of the entropy solution. Let us choose the entropy function $U$ and the entropy fluxes $F_j$, defined as previous chapter, as:

$$\begin{aligned} U(u) &= |u-k|, k \in \mathbb{R}. \\ F_j(u) &= \operatorname{sgn}(u-k)\ (f_j(u) - f_j(k))\,, 1 \le j \le n. \end{aligned}$$

**Lemma 4.0.1:** An entropy solution of the given equation (3.1)-(3.2) satisfies:

$$\int_0^\infty \int_{\mathbb{R}^n} \{|u-k|\,\frac{\partial \phi}{\partial t} + \operatorname{sgn}(u-k)\,\sum_{j=1}^n (f_j(u) - f_j(k))\,\frac{\partial \phi}{\partial x_j}\}\,d\mathbf{x}\,dt \ge 0 \tag{4.1}$$

for all $k \in \mathbb{R}$ and any function $\phi \in C_0^\infty(\mathbb{R}^n \times (0,\infty)), \phi \ge 0$.

**Proof :-** Let, $G$ be a $C^\infty$ function such that,

$$G(x) = |x|, |x| \ge 1 \text{ with } G'' \ge 0.$$

Set

$$G_\epsilon(x) = \epsilon\, G\left(\frac{(x-k)}{\epsilon}\right)$$

so that $G_\epsilon(x) \to |x-k|$ as $\epsilon \to 0$. Moreover, since $G_\epsilon$ is a convex smooth function, we get

$$\frac{\partial}{\partial t} G_\epsilon(u) + \operatorname{div} \mathbf{F}_\epsilon(u) \le 0$$

$$\text{where } \mathbf{F}_\epsilon(u) = \int_k^u G'_\epsilon(v)\,\mathbf{f}'(v)\,dv$$

So, for all $\phi \in C_0^\infty(\mathbb{R}^n \times (0,\infty)), \phi \geq 0$,

$$\int_0^\infty \int_{\mathbb{R}^n} \left\{ G_\epsilon(u)\,\frac{\partial \phi}{\partial t} + \sum_{j=1}^{n} F_{\epsilon j}(u)\,\frac{\partial \phi}{\partial x_j} \right\} d\mathbf{x}\,dt \geq 0 \tag{4.2}$$

Now, $G'_\epsilon(v)$ approximates the function $\operatorname{sgn}(v-k)$, so that,

$$\mathbf{F}_\epsilon(v) \to \mathbf{f}(v) - \mathbf{f}(k), \text{ if } v \geq k.$$

$$\mathbf{F}_\epsilon(v) \to \mathbf{f}(k) - \mathbf{f}(v), \text{ if } v < k.$$

Now, since $\phi$ has a compact support, taking limit in (4.2) as $\epsilon \to 0$, we get,

$$\int_0^\infty \int_{\mathbb{R}^n} \{|u-k|\,\frac{\partial \phi}{\partial t} + \operatorname{sgn}(u-k)\,\sum_{j=1}^{n} (f_j(u) - f_j(k))\,\frac{\partial \phi}{\partial x_j}\}\,d\mathbf{x}\,dt \geq 0$$

This completes the proof.

§ Now we shall state couple of results before going to the main result.

**Lemma 4.0.2:** Let, $u, v \in L^\infty(\mathbb{R}^n \times (0,\infty))$ be two solutions of (3.14), which satisfy the entropy condition in Lemma-4.0.1. Then we have,

$$\frac{\partial}{\partial t}(|u-v|) + \operatorname{div}(\operatorname{sgn}(u-v))\,(\mathbf{f}(u) - \mathbf{f}(v)) \leq 0$$

in the sense of distribution on $\mathbb{R}^n \times (0,\infty)$.

**Lemma 4.0.3:** Assume $f$ satisfies the Lipschitz condition

$$|f(u) - f(v)| \leq M\,|u-v|, \forall u, v \in \Gamma_R = \{u \in \mathbb{R} : |u| \leq R\}.$$

Then the function $(u,v) \mapsto \operatorname{sgn}(u-v)\,(f(u) - f(v))$ satisfies the Lipschitz condition:

$$|\operatorname{sgn}(u-v)(f(u)-f(v)) - \operatorname{sgn}(u*-v*)(f(u*)-f(v*))| \leq M\,(|u-u*| + |v-v*|),$$

for all $u, u*, v, v* \in \Gamma_R$.

**Proof :-** Assume $g(\xi,\omega) = \operatorname{sgn}(\xi-\omega)(f(\xi) - f(\omega))$.

We have for all $\omega \in \Gamma_R$,

$$\begin{aligned} D_\xi\, g(\xi,\omega) &= \operatorname{sgn}(\xi-\omega)\, f'(\xi), \text{ for almost all } \xi \in \Gamma_R \\ \text{and similarly } D_\omega\, g(\xi,\omega) &= -\operatorname{sgn}(\xi-\omega)\, f'(\omega), \text{ for almost all } \omega \in \Gamma_R \end{aligned}$$

Therefore,

$$\begin{aligned}
|\text{sgn}(u-v)(f(u)-f(v)) - \text{sgn}(u*-v*)(f(u*)-f(v*))| &= |g(u,v)-g(u*,v*)| \\
\leq |(u-u*)\,D_\xi\,g(u*,v*)| &+ |(v-v*)\,D_\omega\,g(u*,v*)| \\
&\leq M\,(|u-u*|+|v-v*|)
\end{aligned}$$

Hence the result follows.

**Theorem 4.0.4 [Kruzkov]** Let $u$ and $v$ be two entropy solutions of (3.1)-(3.2) associated with the initial data $u_0$ and $v_0$, both belonging to $L^\infty(\mathbb{R}^n)$, such that $u, v \in L^\infty(\mathbb{R}^n \times (0,\infty)) \cap \mathcal{B}(0,T; L^1_{loc}(\mathbb{R}^n))$ for all $T > 0$. Then, setting

$$M = \max\left\{|f'(\xi)| : |\xi| \leq \max\left\{\|u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}, \|v\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\right\}\right\},$$

we have,

$$\int_{|\mathbf{x}|\leq R} |u(\mathbf{x},t)-v(\mathbf{x},t)|\,d\mathbf{x} \leq \int_{|\mathbf{x}|\leq R+Mt} |u_0(\mathbf{x})-v_0(\mathbf{x})|\,d\mathbf{x} \tag{4.3}$$

**Proof :-** Given two positive numbers $R$ and $T$, we integrate the inequality in Lemma-4.0.2 on the set:

$$D_{R,T} = \{(\mathbf{x},t) \in \mathbb{R}^n \times (0,\infty) : |\mathbf{x}| \leq R + M(T-t), t \in [0,T]\}$$

Thus we get,

$$\begin{aligned}
\int_{|\mathbf{x}|\leq R} |u(\mathbf{x},T)-v(\mathbf{x},T)|\,d\mathbf{x} &- \int_{|\mathbf{x}|\leq R+MT} |u_0(\mathbf{x})-v_0(\mathbf{x})|\,d\mathbf{x} \\
&+ \int_{\Sigma_{R,T}} \{|u-v|\,n_t + \text{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v)).n_{\mathbf{x}}\}\,d\sigma \leq 0
\end{aligned}$$

where, $\Sigma_{R,T} = \{(\mathbf{x},t) \in \mathbb{R}^n \times (0,\infty) : |\mathbf{x}| = R + M(T-t), t \in [0,T]\}$ is the lateral surface of $D_{R,T}$ and $n = (n_{\mathbf{x}}, n_t)$ is the unit outward normal to $\Sigma_{R,T}$.
Since, $|\text{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v)).n_{\mathbf{x}}| \geq -M\,|u-v||n_{\mathbf{x}}|$, we get,

$$|u-v|\,n_t + \text{sgn}(u-v)(\mathbf{f}(u)-\mathbf{f}(v)).n_{\mathbf{x}} \geq |u-v|\,n_t - M|u-v||n_{\mathbf{x}}| = 0$$

on $\Sigma_{R,T}$, so that the result holds.

**Corollary 4.0.5 :-** Assume $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathrm{BV}(\mathbb{R}^n)$ and $\mathbf{f}$ is a $C^1$ function. Then the entropy solution $u$ of problem (3.1)-(3.2) given by Theorem-3.4.4 is unique.

✠ **Remarks:-** The entropy solution depends continuously on the initial condition $u_0$. Also, there is a more general result, which says that, if $u_0 \in L^\infty(\mathbb{R}^n)$, then, the given problem (3.1)-(3.2) has a unique entropy solution $u \in L^\infty(\mathbb{R}^n \times (0,T))$ with

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)}, \quad \text{for almost all } t \geq 0.$$

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