A Dirichlet-Neumann Waveform Relaxation Method for Many Subdomains

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Abstract

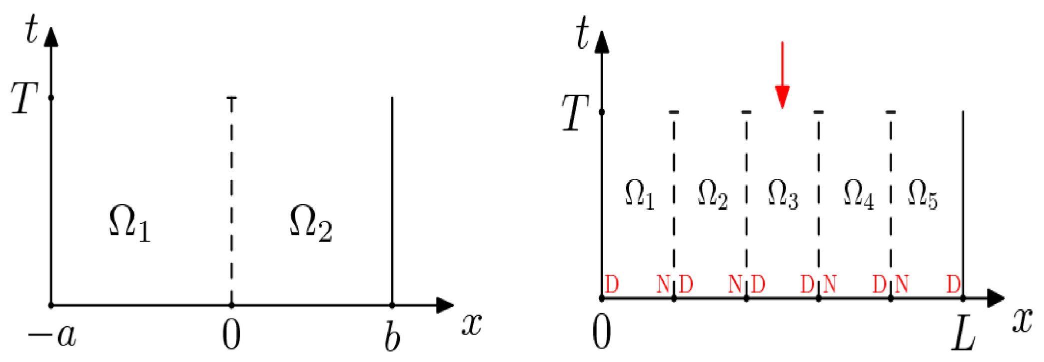
WE propose a generalization of the Dirichlet-Neumann waveform relaxation method to many subdomains in 1D. This method is based on a non-overlapping spatial domain decomposition, and the iteration involves subdomain solves in space-time with interface conditions of either the Dirichlet or Neumann type, depending on the direction of communication. Using a Laplace transform argument, we show that for a particular relaxation parameter, we get superlinear convergence when we consider finite time windows. We also present numerical experiments comparing the DNWR method to the Schwarz Waveform Relaxation method with overlap for two subdomains.

1. Dirichlet-Neumann Waveform Relaxation Method

WE consider the following heat equation on a bounded domain $\Omega \subset \mathbb{R}$, $0 < t < T$ as our guiding example,

$$\left.\begin{aligned} \partial_t u - \partial_{xx} u &= f(x,t), \quad x \in \Omega, 0 < t < T, \\ u(x,0) &= u_0(x), \quad x \in \Omega, \\ u(x,t) &= g(x,t), \quad x \in \partial\Omega, 0 < t < T. \end{aligned}\right\} \qquad (1)$$

We analyze and compare the Dirichlet-Neumann Waveform Relaxation (DNWR) method respectively for two subdomains and many subdomains in one spatial dimension.



(a) Two subdomains (b) Many subdomains

Figure 1: *Decomposition of the domain*

Two subdomains : To define the DNWR method for two subdomains on the domain $(-a, b) \times (0, T)$, we decompose the spatial domain Ω into two non-overlapping subdomains $\Omega_1 = (-a, 0)$ and $\Omega_2 = (0, b)$ as Figure 1(a). The DNWR algorithm for (1) consists of the following steps : given an initial guess $h^0(t), t \in (0, T)$ along the interface $\Gamma = \{x = 0\}$, for $k = 1, 2, \ldots$ do

$$\begin{cases} \partial_t u_1^k - \partial_{xx} u_1^k = f, \\ u_1^k(x,0) = u_0(x), \\ u_1^k(-a,t) = g(-a,t), \\ u_1^k(0,t) = h^{k-1}(t), \end{cases} \quad \begin{cases} \partial_t u_2^k - \partial_{xx} u_2^k = f, \\ u_2^k(x,0) = u_0(x), \\ \partial_x u_2^k(0,t) = \partial_x u_1^k(0,t), \\ u_2^k(b,t) = g(b,t), \end{cases} \qquad (2)$$

with the updating condition

$$h^k(t) = \theta u_2^k(0,t) + (1-\theta) h^{k-1}(t), \qquad (3)$$

θ being a positive relaxation parameter and $u_i \in \Omega_i \times (0, T)$, for $i = 1, 2$.

Many subdomains : To illustrate the generalization to many subdomains, let us consider a decomposition of $\Omega = (0, L)$ into five non-overlapping subdomains $\Omega_i = (x_{i-1}, x_i)$, $i = 1, \ldots, 5$ with $x_0 = 0, x_5 = L$ as Figure 1(b). The general N-subdomain case is treated analogously. We define a new version of DNWR iteration for multiple subdomains as proposed in [2]. Given initial Dirichlet data $\{g_i^0(t)\}_{i=1}^4$ and denoting $\{w_i^k(t)\}_{i=1}^4$ for $k = 1, 2 \ldots$ as Neumann traces along the 4 interfaces, we have for $t \in (0, T)$

$$\begin{cases} \partial_t u_3^k - \partial_{xx} u_3^k = f, & x \in \Omega_3, \\ u_3^k(x,0) = u_0(x), & x \in \Omega_3, \\ u_3^k(x_2,t) = g_2^{k-1}(t), \\ u_3^k(x_3,t) = g_3^{k-1}(t). \end{cases}$$

and then

$$\begin{cases} \partial_t u_2^k - \partial_{xx} u_2^k = f, & x \in \Omega_2, \\ u_2^k(x,0) = u_0(x), & x \in \Omega_2, \\ u_2^k(x_1,t) = g_1^{k-1}(t), \\ \partial_x u_2^k(x_2,t) = w_3^k(t). \end{cases} \begin{cases} \partial_t u_4^k - \partial_{xx} u_4^k = f, & x \in \Omega_4, \\ u_4^k(x,0) = u_0(x), & x \in \Omega_4, \\ -\partial_x u_4^k(x_3,t) = w_3^k(t), \\ u_4^k(x_4,t) = g_4^{k-1}(t). \end{cases} \qquad (4)$$

$$\begin{cases} \partial_t u_1^k - \partial_{xx} u_1^k = f, & x \in \Omega_1, \\ u_1^k(x,0) = u_0(x), & x \in \Omega_1, \\ u_1^k(0,t) = g(0,t), \\ \partial_x u_1^k(x_1,t) = w_2^k(t). \end{cases} \begin{cases} \partial_t u_5^k - \partial_{xx} u_5^k = f, & x \in \Omega_5, \\ u_5^k(x,0) = u_0(x), & x \in \Omega_5, \\ -\partial_x u_5^k(x_4,t) = w_4^k(t), \\ u_5^k(L,t) = g(L,t). \end{cases}$$

The update conditions are : for $i = 1, 2$ and $j = 3, 4$

$$\begin{cases} g_i^k(t) = \theta u_i^k(x_i,t) + (1-\theta) g_i^{k-1}(t); \; w_i^k(t) = \partial_x u_{i+1}^k(x_i,t), \\ g_j^k(t) = \theta u_{j+1}^k(x_j,t) + (1-\theta) g_j^{k-1}(t); \; w_j^k(t) = -\partial_x u_j^k(x_j,t). \end{cases} \qquad (5)$$

Remark 1.1. *It suffices by linearity to consider the homogeneous problem, $f(x,t) = 0 = g(x,t) = u_0(x)$ in (1), and see how $h^k(t)$ in (3) and $g_i^k(t)$ in (5) go to zero as $k \to \infty$.*

2. Convergence Results Using Laplace Transforms

Two subdomains case

THE following results show that the asymptotic convergence rate of the DNWR algorithm in a finite time window is superlinear for a particular value of the relaxation parameter. We give two estimates of the error for $\theta = 1/2$.

Theorem 2.1. ***(Convergence of DNWR, for $a > b$)*** *Let $\theta = 1/2$. Then the error of the DNWR algorithm for two subdomains, with $a > b$, satisfies for $t \in (0, \infty)$*

$$\| h^k \|_{L^\infty(0,\infty)} \leq \left(\frac{a-b}{2a}\right)^k \| h^0 \|_{L^\infty(0,\infty)} .$$

For a finite time interval $(0, T)$, the DNWR converges superlinearly with the estimate

$$\| h^k \|_{L^\infty(0,T)} \leq \left(\frac{a-b}{a}\right)^k erfc\left(\frac{kb}{2\sqrt{T}}\right) \| h^0 \|_{L^\infty(0,T)} .$$

Theorem 2.2. ***(Convergence of DNWR, for $a < b$)*** *Let $\theta = 1/2$. Then the error of the DNWR algorithm for the two subdomains, with $a < b$, satisfies for $t \in (0, \infty)$*

$$\| h^{2k} \|_{L^\infty(0,\infty)} \leq \left(\frac{b-a}{2a}\right)^{2k} \| h^0 \|_{L^\infty(0,\infty)} .$$

For a finite time interval $(0, T)$, the DNWR converges superlinearly with the estimate

$$\| h^{2k} \|_{L^\infty(0,T)} \leq \left(\frac{\sqrt{2}}{1 - e^{-\frac{(2k+1)a^2}{T}}}\right)^{2k} e^{-k^2 a^2/T} \| h^0 \|_{L^\infty(0,T)} .$$

Remark 2.3. *For the symmetric case $(a = b)$ we get convergence in two steps for $\theta = 1/2$. For non-symmetric cases, $a \neq b$, we get linear convergence for all θ other than $1/2$.*

Multiple subdomains case

WE prove convergence of the DNWR for general N (odd) equi-spaced subdomains. Extending the algorithm (4)-(5) in a natural way, we apply Laplace transform to get expressions of the updating conditions in the Laplace space. We denote η as Laplace variable. For the N-subdomain case $(N = 2p + 1)$, we can write the updating terms as

$$\begin{pmatrix} \hat{g}_1^{k+1} \\ \hat{g}_2^{k+1} \\ \vdots \\ \vdots \\ \hat{g}_{N-2}^{k+1} \\ \hat{g}_{N-1}^{k+1} \end{pmatrix} = C \begin{pmatrix} \hat{g}_1^{k} \\ \hat{g}_2^{k} \\ \vdots \\ \vdots \\ \hat{g}_{N-2}^{k} \\ \hat{g}_{N-1}^{k} \end{pmatrix}, \text{ with}$$

$$C = \begin{bmatrix} \frac{1}{2c^2} & -\frac{s^2}{2c^3} & \cdots & -\frac{s^2}{2c^p} & -\frac{1}{2c^{p-1}} & \frac{1}{2c^p} & 0 & 0 & 0 & 0 \\ \frac{1}{2c} & \frac{1}{2c^2} & \ddots & \ddots & -\frac{1}{2c^{p-2}} & \frac{1}{2c^{p-1}} & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \frac{1}{2c} & \frac{1}{2c^2} & -\frac{1}{2c} & \frac{1}{2c^2} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & \frac{1}{2c} & 0 & \frac{1}{2c} & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{1}{2c} & 0 & \frac{1}{2c} & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{1}{2c^2} & -\frac{1}{2c} & \frac{1}{2c^2} & \frac{1}{2c} & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2c^{p-1}} & -\frac{1}{2c^{p-2}} & \ddots & \ddots & \frac{1}{2c^2} & \frac{1}{2c} \\ 0 & 0 & 0 & 0 & \frac{1}{2c^p} & -\frac{1}{2c^{p-1}} & -\frac{s^2}{2c^p} & \cdots & -\frac{s^2}{2c^3} & \frac{1}{2c^2} \end{bmatrix}.$$

denoting $H = L/N, c = \cosh(H\sqrt{\eta})$ and $s = \sinh(H\sqrt{\eta})$. We then use various inequality properties and kernel estimates from [1] to have convergence estimates.

Theorem 2.4. ***(Convergence of DNWR for N subdomains)*** *Suppose N is odd. Then for $\theta = 1/2$ and for a finite time interval $(0, T)$, the DNWR algorithm for N equi-length subdomains with common size H, converges superlinearly with the estimate*

$$\max_{1 \leq i \leq N-1} \| g_i^k \|_{L^\infty(0,T)} \leq \psi^k erfc\left(\frac{kH}{2\sqrt{T}}\right) \max_{1 \leq i \leq N-1} \| g_i^0 \|_{L^\infty(0,T)},$$

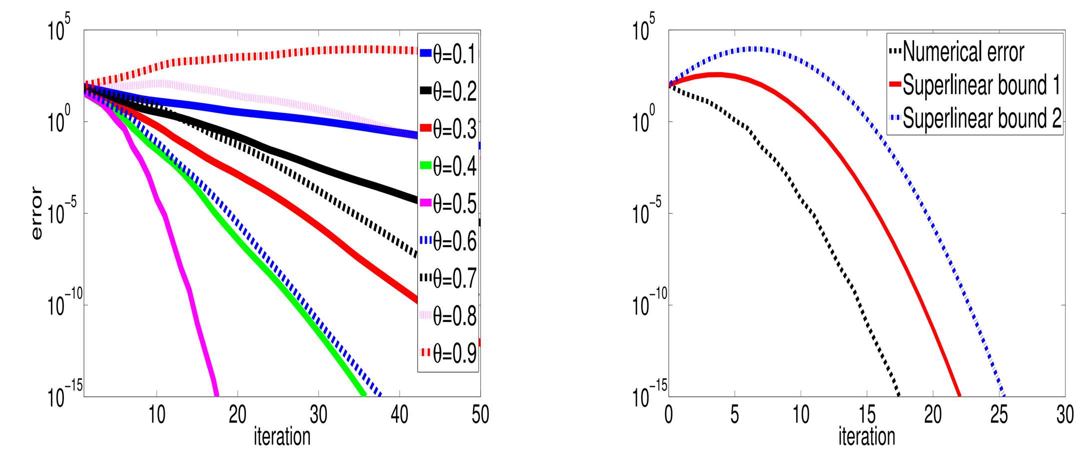
where $\psi(N, T) = \min\{(N-2), \phi(N, T)\}$

with $\phi(N, T) = 2erfc\left(\frac{H}{2\sqrt{T}}\right) + \sum_{i=0}^{(N-3)/2} 2^{i+1} erfc\left(\frac{iH}{2\sqrt{T}}\right)$.

Remark 2.5. *In case of even N, we treat in a similar way as above the first $N - 1$ subdomains, keeping the last one aside. Then for the last subdomain, we apply a Neumann transmission condition along the interface and a Dirichlet boundary condition along the physical boundary.*

3. Numerical Experiments

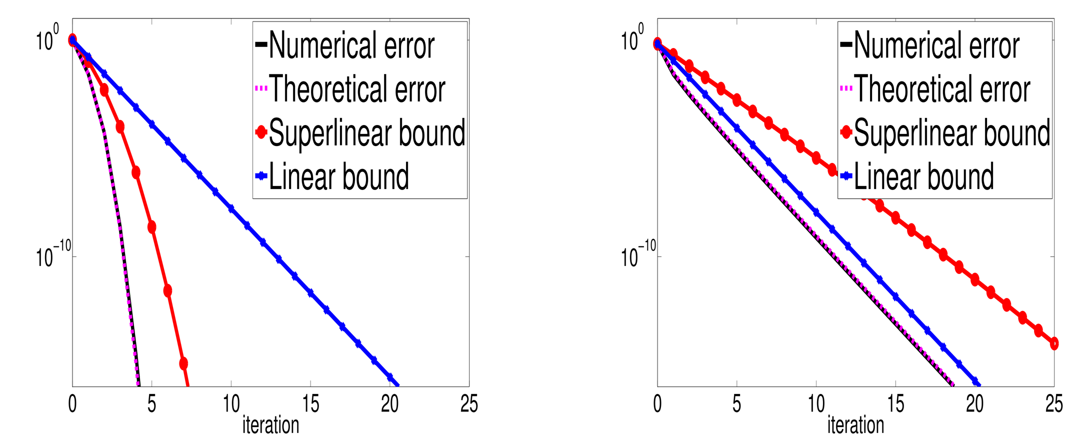
WE split the spatial domain $[0, 5]$ into five non-overlapping subdomains of equal length to apply the DNWR iteration for a small time $T = 2$. We discretize (1) using centered finite differences in space and backward Euler in time with step size $\Delta x = 1/50$ and $\Delta t = 1/250$. Among all the parameters we get a faster superlinear convergence for $\theta = 0.5$. Figure 2(b) gives the convergence estimates with ψ and ϕ as shown in Theorem 2.4 in case of $\theta = 0.5$.



(a) Error curves (b) Estimates for $\theta = 0.5$

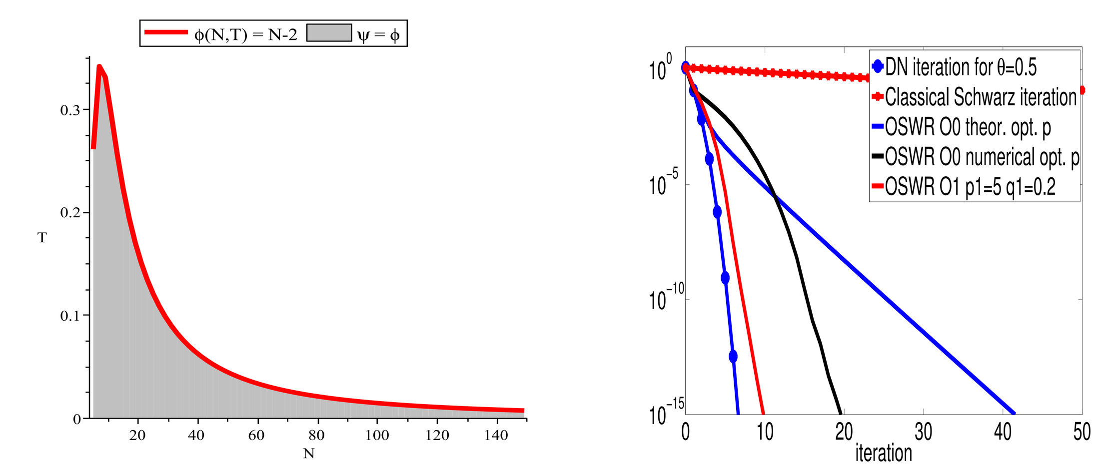
Figure 2: *Convergence for various parameters.*

We now plot the estimates for the convergence rate in case of $\theta = 0.5$ for the two unequal subdomains case. Figure 3 gives a comparison between the theoretical error, numerical error, linear bound and the superlinear bound shown in Theorem 2.1. Observe that the error curves approach the linear bound as T increases.



(a) $T = 2$ (b) $T = 200$

Figure 3: *Convergence estimates, $a = 3, b = 2$.*



(a) Region for $\phi(N, T) < N - 2$ (b) DNWR and OSWR

Figure 4: *Comparison of convergence estimates.*

Finally to compare the performance of the proposed algorithm with Schwarz Waveform Relaxation with overlap, we employ an overlap of length $2\Delta x$. We observe faster convergence in 2-subdomains DNWR than in overlapping Schwarz WR in Figure 4(b).

Conclusions and Further Works

WE proved convergence of the DNWR algorithm for two as well as many subdomains. For two unequal subdomains and for a particular choice of the relaxation parameter, we presented a superlinear error estimate. We proved superlinear convergence for that particular θ and for many equi-spaced subdomains. We are working on a adaptation of the algorithm to higher dimensions.

References

[1] Gander, M.J., Kwok, F., Mandal, B.C. :*Dirichlet-Neumann and Neumann-Neumann Waveform Relaxation Methods for the Heat Equation*, Submitted in SINUM.

[2] Funaro, D., Quarteroni, A., Zanolli, P. : *An Iterative Procedure with Interface Relaxation for Domain Decomposition Methods*, SIAM J. Numer. Anal.,1988.