Dirichlet-Neumann Method for the Time-Dependent Problems

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Abstract

We present a waveform relaxation version of the Dirichlet-Neumann method for the heat and wave equations. Like the Dirichlet-Neumann method for steady problems, the method is based on a non-overlapping spatial domain decomposition, and the iteration involves subdomain solves with Dirichlet boundary conditions followed by subdomain solves with Neumann boundary conditions. However, each subdomain problem is now in space and time, and the interface conditions are also time-dependent. We call the algorithm the Dirichlet-Neumann Waveform Relaxation (DNWR) method. Using a Laplace transform argument, we show for the heat equation that when we consider finite time intervals, the DNWR method converges superlinearly for a particular choice of the relaxation parameter. For the wave equation we prove convergence in finite number of steps for a particular parameter. We also present numerical experiments, comparing the DNWR method to the Schwarz Waveform Relaxation method with overlap.

1. Dirichlet-Neumann Waveform Relaxation Method

We consider the following heat and wave equations on a bounded domain $\Omega \subset \mathbb{R}$, $0 < t < T$ as our guiding examples,

$$\left.\begin{aligned} \partial_t u - \partial_{xx}u &= f_1(x,t), & x\in\Omega, 0<t<T,\\ u(x,0) &= u_0(x), & x\in\Omega,\\ u(x,t) &= g_1(x,t), & x\in\partial\Omega, 0<t<T. \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} \partial_{tt} u - c^2\partial_{xx}u &= f_2(x,t), & x\in\Omega, 0<t<T,\\ u(x,0) &= w_0(x), & x\in\Omega,\\ u_t(x,0) &= \bar{w}_0(x), & x\in\Omega,\\ u(x,t) &= g_2(x,t), & x\in\partial\Omega, 0<t<T. \end{aligned}\right\} \quad (2)$$

To define the Dirichlet-Neumann Waveform Relaxation method for the model problems (1)-(2) on the domain $(-b,a)\times(0,T)$, we decompose the spatial domain, $\Omega=(-b,a)$ into two non-overlapping subdomains $\Omega_1=(-b,0)$ and $\Omega_2=(0,a)$, for $0<a,b<\infty$.

The DNWR algorithm for the heat equation consists of the following steps : given an initial guess $h_1^0(t), t\in(0,T)$ along the interface $\Gamma=\{x=0\}$, for $k=0,1,2,\dots$ do

$$\begin{cases} \partial_t u_1^{k+1} - \partial_{xx}u_1^{k+1} = f_1(x,t), & x\in\Omega_1,\\ u_1^{k+1}(x,0) = u_0(x), & x\in\Omega_1,\\ u_1^{k+1}(-b,t) = g_1(-b,t), & \\ u_1^{k+1}(0,t) = h_1^k(t), & \end{cases} \quad (3)$$

and

$$\begin{cases} \partial_t u_2^{k+1} - \partial_{xx}u_2^{k+1} = f_1(x,t), & x\in\Omega_2,\\ u_2^{k+1}(x,0) = u_0(x), & x\in\Omega_2,\\ \partial_x u_2^{k+1}(0,t) = \partial_x u_1^{k+1}(0,t), & \\ u_2^{k+1}(a,t) = g_1(a,t), & \end{cases} \quad (4)$$

with the updating condition

$$h_1^{k+1}(t) = \theta u_2^{k+1}(0,t) + (1-\theta)h_1^k(t), \quad (5)$$

θ being a positive relaxation parameter. We define an analogous iteration for the wave equation with an initial guess $h_2^0(t), t\in(0,T)$ along the interface.

Goal : To study how the error $h_i^k(t)-u(0,t)$ converges to zero for $i=1,2$. In fact, it suffices by linearity to consider the homogeneous problems, $f_i(x,t)=0=g_i(x,t), w_0(x)=0=\bar{w}_0(x)=u_0(x)$ in (1)-(2), and to examine how $h_i^k(t)$ goes to zero as $k\to\infty$.

2. Convergence Results Using Laplace Transforms

Heat Equation

The following results show that the asymptotic convergence rate of the DNWR algorithm in a finite time window is superlinear for a particular value of the relaxation parameter. We give two estimates of the error for $\theta=1/2$.

Theorem 2.1. ***(Convergence of DNWR, for $b\geq a$)*** *Let $\theta=1/2$. Then the error of the DNWR algorithm for two subdomains, with $b\geq a$, satisfies for $t\in(0,\infty)$*

$$\| h_1^k \|_{L^\infty(0,\infty)} \leq \left(\frac{b-a}{2b}\right)^k \| h_1^0 \|_{L^\infty(0,\infty)}.$$

For a finite time interval $(0,T)$, the DNWR converges superlinearly with the estimate

$$\| h_1^k \|_{L^\infty(0,T)} \leq \left(\frac{b-a}{b}\right)^k \operatorname{erfc}\left(\frac{ka}{2\sqrt{T}}\right) \| h_1^0 \|_{L^\infty(0,T)}.$$

Theorem 2.2. ***(Convergence of DNWR, for $a\geq b$)*** *Let $\theta=1/2$. Then the error of the DNWR algorithm for the two subdomains, with $a\geq b$, satisfies for $t\in(0,\infty)$*

$$\| h_1^{2k} \|_{L^\infty(0,\infty)} \leq \left(\frac{a-b}{2b}\right)^{2k} \| h_1^0 \|_{L^\infty(0,\infty)}.$$

For a finite time interval $(0,T)$, the DNWR converges superlinearly with the estimate

$$\| h_1^{2k} \|_{L^\infty(0,T)} \leq \left\{\frac{\sqrt{2}}{1-e^{-\frac{2k+1}{\sigma}}}\right\}^{2k} e^{-k^2/\sigma} \| h_1^0 \|_{L^\infty(0,T)},$$

where $\sigma=T/b^2$.

Remark 2.3. *For the symmetric case $(a=b)$ we get convergence in two steps for $\theta=1/2$. For non-symmetric cases, $a\neq b$, we observe linear convergence for all values of the parameter other than $1/2$.*

Reason behind superlinearity : After applying the Laplace transform, we get the updating condition (5) for $k=1,2,\dots$

$$\hat{h}_1^k(s) = \begin{cases} \{q(\theta)-\theta G(s)\}^k \hat{h}^0(s), & \theta\neq 1/2\\ (-1)^k 2^{-k} G^k(s)\hat{h}^0(s), & \theta=1/2, \end{cases} \quad (6)$$

where $q(\theta)=1-2\theta$ and $G(s):=\frac{\sinh((a-b)\sqrt{s})}{\cosh(a\sqrt{s})\sinh(b\sqrt{s})}$. Now if we define the kernel, $F_k(t)=\mathcal{L}^{-1}\left\{G^k(s)\right\}$, then Figure 1 of the kernel indicates superlinear convergence for $\theta=1/2$, because we see that the curves shift to the right and at the same time the peak decreases as k increases. So, if one only considers a small time window, the peak will eventually exit the time window for k large enough and its contribution will be vanishingly small in the expression (6).

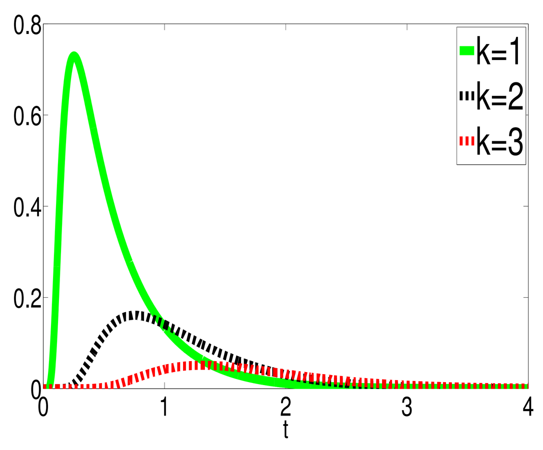


Figure 1: *$F_k(t), k=1,2,3$.*

For $\theta\neq 1/2$, $q(\theta)$ will dominate in the expression (6) to produce linear convergence for other values of θ.

Wave Equation

We prove two-step convergence of the DNWR iteration for the wave equation in the symmetric case for a particular choice of the relaxation parameter. For a sufficiently small time window, we show linear convergence for $\theta=1/2$.

Theorem 2.4. ***(Convergence in the symmetric case)*** *For the symmetric case, $a=b$, the DNWR algorithm converges linearly for $0<\theta<1$. For $\theta=1/2$, it converges to the exact solution in two iterations, independent of the size of the time window.*

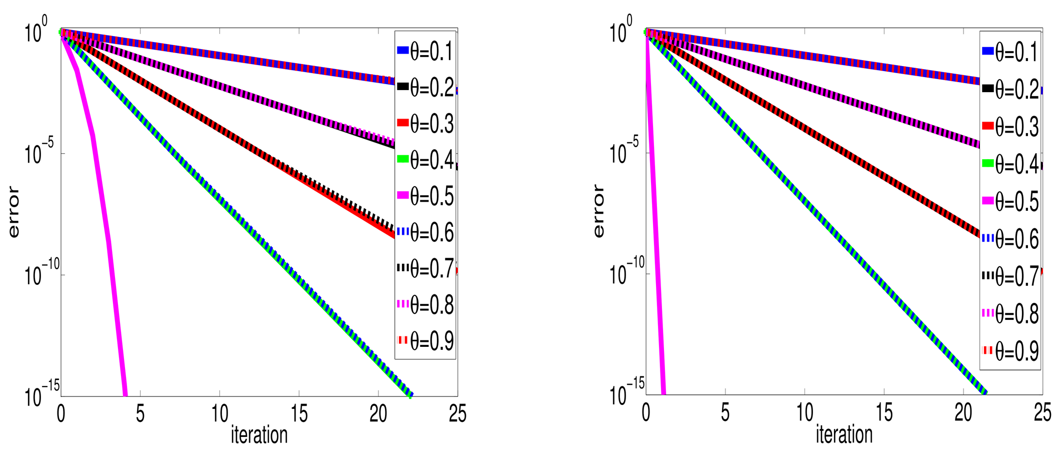
Theorem 2.5. ***(Convergence for finite time)*** *Let $\theta=1/2$. Then the DNWR algorithm converges in at most $m+1$ iterations for two subdomains of lengths a and b, if the time T satisfies*

$$T/m \leq 2\min\{a/c, b/c\}.$$

Remark 2.6. *In particular, for non-symmetric subdomains the error of the DNWR method vanishes in two iterations in case of $\theta=1/2$, if $T\leq 2\min\{a/c,b/c\}$. We observe linear convergence for other values of the relaxation parameter.*

Numerical Experiments

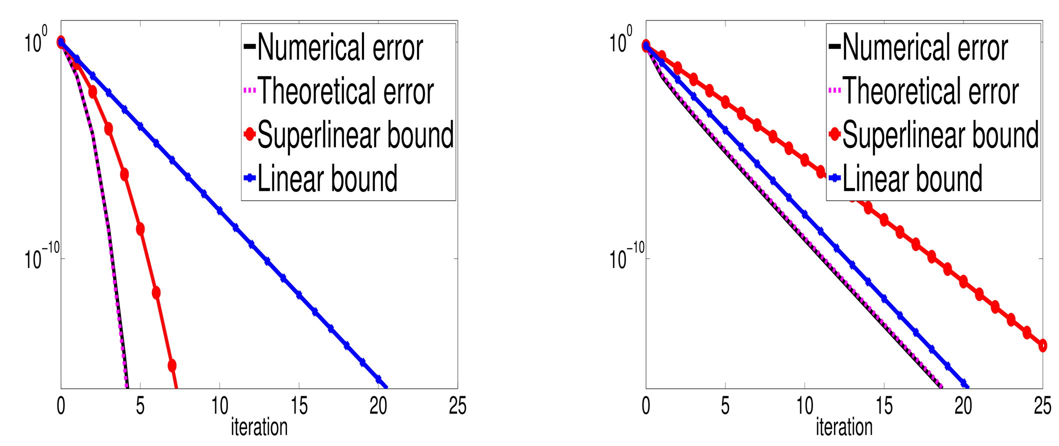
We split the spatial domain $[-3,2]$ into two non-overlapping subdomains $[-3,0]$ and $[0,2]$ to apply the DNWR iteration for a small time domain $T=2$. For the heat equation, we get linear convergence for all the parameters, except for $\theta=0.5$ which corresponds to superlinear convergence. Similarly for the wave equation we observe linear convergence for all the parameters, except for $\theta=0.5$ that corresponds to convergence in two iterations. The numerical results are similar when the Neumann subdomain is larger than the Dirichlet one.



(a) Heat equation (b) Wave equation

Figure 2: *Convergence for various parameters, $b\geq a$.*

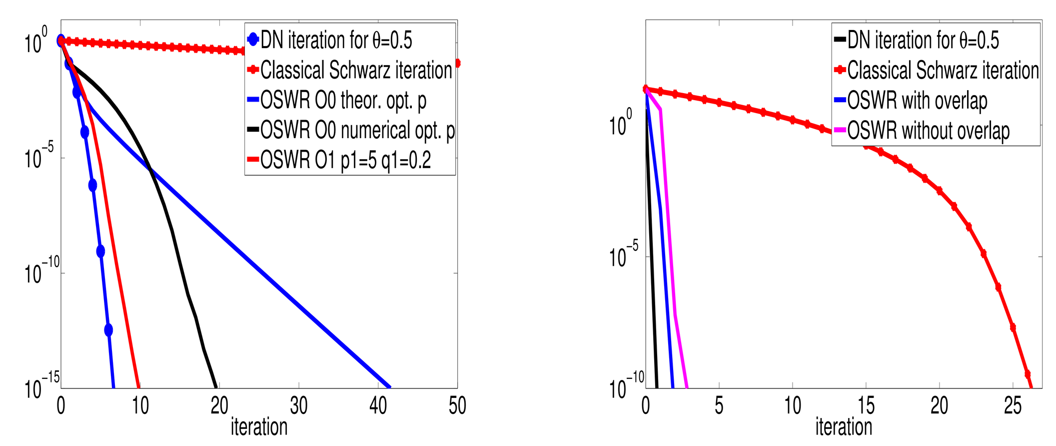
We now plot the estimates for the convergence rate in case of $\theta=0.5$ for the heat equation. Figure 3 gives a comparison between the theoretical error, numerical error, linear bound and the superlinear bound shown in Theorem 2.1. Observe that the error curves approach the linear bound as T increases.



(a) $T=2$ (b) $T=200$

Figure 3: *Bounds for various times, $b\geq a$.*

Finally to compare the performance of the proposed algorithm with Schwarz Waveform Relaxation with overlap, we employ an overlap of length $2\Delta x$, where $\Delta x=1/50$ and $\Delta t=1/250$. We have faster convergence in DNWR than in overlapping Schwarz WR for both problems.



(a) Heat equation (b) Wave equation

Figure 4: *Comparison of DNWR with OSWR.*

Conclusions and Further Works

We proved convergence of the DNWR algorithm in the symmetric case for both equations. For unequal subdomain lengths and for a particular choice of the relaxation parameter, we presented a superlinear error estimate for the heat equation. For the wave equation we proved linear convergence for a particular θ in finite time windows. We are working on a generalization of the algorithm to many subdomains and adaptation to higher dimensions as well.

References

[1] Gander, M.J., Stuart, A.M. : *Space-Time Continuous Analysis of Waveform Relaxation for the Heat Equation*, SIAM J. for Sci. Comput., 2014-2031, 1998.
[2] Giladi, E., Keller, H. : *Space Time Domain Decomposition for Parabolic Problems*, Numer. Math., 279-313, 2002.
[3] Oberhettinger, F., Badii, L. : *Tables of Laplace Transforms*, Springer-Verlag, 1973.

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