Non-overlapping Domain Decomposition Algorithm for time dependent problem

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Abstract

THE purpose of the work is to present and analyze a new class of Non-overlapping Domain Decomposition methods for parabolic problems in one space dimension. We mainly focus on two kinds of methods. In the first method, the heat equation in bounded space domain is solved iteratively with a Dirichlet-Neumann preconditioner. The second method is direct and similar to iterative methods with a Neumann-Neumann preconditioner.

Iterative Substructuring Algorithms

Dirichlet-Neumann Algorithms

WE consider the heat equation in $\Omega \times (0,\infty)$, Ω being (0,1) :

$$\left.\begin{aligned} u_t - u_{xx} &= 0,\ x \in \Omega, t > 0 \\ u(0,t) &= g_1(t),\ t > 0 \\ u(1,t) &= g_2(t),\ t > 0 \\ u(x,0) &= u_0(x),\ x \in \Omega \end{aligned}\right\} \qquad (1)$$

Supposing $\alpha \in (0,1)$, we decompose the domain Ω as $\Omega_1 = (0,\alpha)$ and $\Omega_2 = (\alpha, 1)$ and $\Gamma = \{x = \alpha\}$.
The generalization of the Dirichlet-Neumann method for the problem (1) gives :
Given λ^0, solve for each $k \geq 0$,

$$\begin{cases} \frac{\partial u_1^{k+1}}{\partial t} = u_{1,xx}^{k+1},\ x \in \Omega_1, t > 0 \\ u_1^{k+1}(0,t) = g_1(t),\ t > 0 \\ u_1^{k+1}(\alpha,t) = \lambda^k,\ t > 0 \\ u_1^{k+1}(x,0) = u_0(x),\ x \in \Omega_1 \end{cases}$$

Then,

$$\begin{cases} \frac{\partial u_2^{k+1}}{\partial t} = u_{2,xx}^{k+1},\ x \in \Omega_2, t > 0 \\ \frac{\partial u_2^{k+1}}{\partial x}\Big|_{x=\alpha} = \frac{\partial u_1^{k+1}}{\partial x}\Big|_{x=\alpha} \\ u_2^{k+1}(1,t) = g_2(t),\ t > 0 \\ u_2^{k+1}(x,0) = u_0(x),\ x \in \Omega_2 \end{cases}$$

with $\lambda^{k+1} = \theta u_{2,\Gamma}^{k+1} + (1-\theta)\lambda^k$.

Consider the parabolic equation in $\mathbb{R} \times (0,\infty)$:

$$\left.\begin{aligned} u_t - u_{xx} &= 0,\ x \in \mathbb{R}, t > 0 \\ u(x,0) &= u_0(x),\ x \in \mathbb{R} \end{aligned}\right\} \qquad (2)$$

In this case, we decompose the domain as $\mathbb{R}^+$ and $\mathbb{R}^-$ and $\Gamma = \{x = 0\}$.
The generalization of the Dirichlet-Neumann method for the problem (1) gives :
Given λ^0,

$$\begin{cases} \frac{\partial u_1^{k+1}}{\partial t} = u_{1,xx}^{k+1},\ x \in \mathbb{R}^-, t > 0 \\ u_1^{k+1}(x,0) = u_0(x),\ x \in \mathbb{R}^- \\ u_1^{k+1} = \lambda^k,\ \text{on } \Gamma \end{cases}$$

Then,

$$\begin{cases} \frac{\partial u_2^{k+1}}{\partial t} = u_{2,xx}^{k+1},\ x \in \mathbb{R}^+, t > 0 \\ u_2^{k+1}(x,0) = u_0(x),\ x \in \mathbb{R}^+ \\ \frac{\partial u_2^{k+1}}{\partial x} = \frac{\partial u_1^{k+1}}{\partial x},\ \text{on } \Gamma \end{cases}$$

with $\lambda^{k+1} = \theta u_{2,\Gamma}^{k+1} + (1-\theta)\lambda^k$, θ being a positive acceleration parameter.
In this case, if we define :

$$e_1^{k+1} := u_1^{k+1} - u_1 \text{ and } e_2^{k+1} := u_2^{k+1} - u_2,$$

then, using Green function method , we get,

$$e_1^{k+1}(x\,,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}} \exp\left(-\frac{x^2}{4(t-\tau)}\right) \lambda^k(\tau)\,d\tau$$

$$e_2^{k+1}(x\,,t) = 2\int_0^t K(x\,,t-\tau)\int_0^\tau \frac{1}{\sqrt{4\pi(\tau-s)^3}}\lambda^k(s)\,ds\,d\tau$$

$$\lambda^{k+1} = \frac{\theta}{2\pi}\int_{\tau=0}^t \frac{1}{\sqrt{(t-\tau)}}\int_{s=0}^\tau \frac{1}{\sqrt{(\tau-s)^3}}\lambda^k(s)\,ds\,d\tau + (1-\theta)\lambda^k$$

where, $K(x,t)$ is the Heat Kernel.

Goal : *We will go for an estimate to*

$$\int_{\tau=0}^t \frac{1}{\sqrt{(t-\tau)}}\int_{s=0}^\tau \frac{1}{\sqrt{(\tau-s)^3}}\lambda^k(s)\,ds\,d\tau.$$

Neumann-Neumann Algorithms

The generalization of the Neumann-Neumann method for the equation (2) gives :
Given λ^0, solve for each $k \geq 0$,

$$\begin{cases} \frac{\partial u_i^{k+1}}{\partial t} = u_{i,xx}^{k+1},\ x \in (\mathbb{R}^- \text{ or } \mathbb{R}^+), t > 0 \\ u_i^{k+1}(x,0) = u_0(x),\ x \in \mathbb{R}^- \text{ or } \mathbb{R}^+ \\ u_i^{k+1} = \lambda^k,\ \text{on } \Gamma \end{cases}$$

and then,

$$\begin{cases} \frac{\partial \psi_i^{k+1}}{\partial t} = \psi_{i,xx}^{k+1},\ x \in (\mathbb{R}^- \text{ or } \mathbb{R}^+), t > 0 \\ \psi_i^{k+1}(x,0) = u_0(x),\ x \in \mathbb{R}^- \text{ or } \mathbb{R}^+ \\ \frac{\partial \psi_i^{k+1}}{\partial x} = \frac{\partial u_1^{k+1}}{\partial x} - \frac{\partial u_2^{k+1}}{\partial x},\ \text{on } \Gamma \end{cases}$$

for i = 1,2 with $\lambda^{k+1} = \lambda^k - \theta(\sigma_1\psi_{1|\Gamma}^{k+1} - \sigma_2\psi_{2|\Gamma}^{k+1})$, σ_1 , σ_2 are positive weights.

Primal and Dual Schur Approximation

Primal Schur

THE equation in (2) is equivalent to :

$$\begin{cases} u_t^j = u_{xx}^j,\ x \in \Omega_j, t > 0 \\ u^j(x,0) = u_0(x),\ x \in \Omega_j \\ u^j = \lambda,\ \text{on } \Gamma \\ \frac{\partial u^1}{\partial x}\Big|_{x=0} = \frac{\partial u^2}{\partial x}\Big|_{x=0} \end{cases}$$

for j = 1,2. Here $\Omega_1 = \mathbb{R}^-$, $\Omega_2 = \mathbb{R}^+$ and $\Gamma = \{x = 0\}$.
Therefore, the solutions u^1 and u^2 have the form :

$$u^1(x\,,t) = \int_{-\infty}^0 G(x,y,t)u_0(y)\,dy - \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}} \exp\left(-\frac{x^2}{4(t-\tau)}\right)\lambda(\tau)\,d\tau$$

$$u^2(x\,,t) = \int_0^\infty G(x,y,t)u_0(y)\,dy + \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}} \exp\left(-\frac{x^2}{4(t-\tau)}\right)\lambda(\tau)\,d\tau$$

where, $G(x,y,t) = K(x-y,t) - K(x+y,t)$
Moreover, we have :

$$\frac{\partial u^1}{\partial x}\Big|_{x=0} = \frac{\partial u^2}{\partial x}\Big|_{x=0}$$

Therefore,

$$2\int_0^t \frac{1}{\sqrt{4\pi(t-\tau)^3}}\lambda(\tau)\,d\tau = \frac{\partial}{\partial x}\left\{\int_{-\infty}^0 G(x,y,t)u_0(y)\,dy - \int_0^\infty G(x,y,t)u_0(y)\,dy\right\}\Big|_{x=0}$$

Now, using a suitable quadrature formula, for a partition $(0, t_1, \ldots, t_n = 1)$, we get the equivalence form of the left hand side as :

$$\sum_{i=0}^j \omega_i L(t_j - t_i)\,\lambda(t_i)$$

Using this form, we get the equivalent matrix form to the above equation as :

$$A\,\lambda = b$$

where $\lambda(t_i) \sim \lambda_i$. Clearly, the matrix A is not symmetric, rather lower triangular.
Goal : *Further Analysis on spectral property of the matrix A.*

Dual Schur

In this case, the equivalent system will be :

$$\begin{cases} u_t^j = u_{xx}^j,\ x \in \Omega_j, t > 0 \\ u^j(x,0) = u_0(x),\ x \in \Omega_j \\ \frac{\partial u^j}{\partial x} = \lambda,\ \text{on } \Gamma \\ u^1\Big|_{x=0} = u^2\Big|_{x=0} \end{cases}$$

for j = 1,2. Therefore, the solutions u^1 and u^2 have the form :

$$u^1(x\,,t) = \int_{-\infty}^0 N(x,y,t)u_0(y)\,dy + 2\int_0^t K(x,t-\tau)\,\lambda(\tau)\,d\tau$$

$$u^2(x\,,t) = \int_0^\infty N(x,y,t)u_0(y)\,dy - 2\int_0^t K(x,t-\tau)\,\lambda(\tau)\,d\tau$$

where, $N(x,y,t) = K(x-y,t) + K(x+y,t)$
Therefore,

$$4\int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}}\lambda(\tau)\,d\tau = \left\{\int_0^\infty N(x,y,t)u_0(y)\,dy - \int_{-\infty}^0 N(x,y,t)u_0(y)\,dy\right\}\Big|_{x=0}$$

We treat it in a similar way as the case of Primal.

Numerical Experiments

WE , now, show an experiment , which solves the generalization of the Dirichlet-Neumann method for the equation (1) numerically. On the half of the domain, iteration takes place with Dirichlet Boundary condition and Neumann condition on the other half.

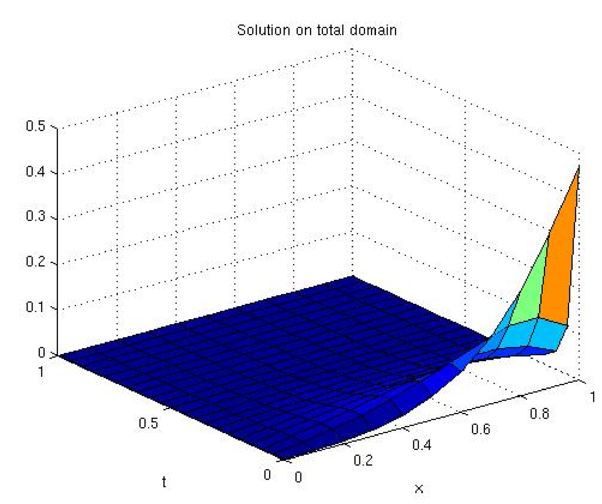

Figure 1: *Solution of Heat equation with Dirichlet-Neumann boundary condition*

Future works

WE will focus on the theory of boundary value problems by Integral Operators and try to do more possible numerical experiments. Also, we will focus on substructuring methods for Parabolic problems. *Thanks to Felix Kwok for his support and help.*

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